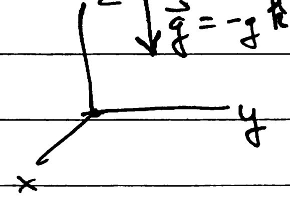
In order to obtain a deeper intuitive feeling for Newton's laws, let's study the trajectory of a particle when subject to various types of forces.

<u>Example 1</u> Consider single particle motion under a constant gravitational force aligned in the z-direction



$$\vec{F} = m\vec{g} = -mg\hat{k}$$

$$m\ddot{\vec{x}} = \vec{F} = -mg\hat{k}$$

$$\Rightarrow \ddot{x} = 0 = \ddot{y} \quad \Rightarrow \quad x = x_0 + v_x t$$
$$y = y_0 + v_y t$$

$$\ddot{z} = -g \quad \Rightarrow \quad z = z_0 + v_z t - \tfrac{1}{2} g t^2$$

In general we desire to predict the position of ~~any~~ the particle ~~at~~ the next instant, given the present position. Since $\vec{x}(t+dt) = \vec{x}(t) + \dot{\vec{x}}(t)\,dt$ we need also to specify $\dot{\vec{x}}(t)$ as well as $\vec{x}(t)$. But to know $\dot{\vec{x}}(t)$ at the following instant we need $\ddot{\vec{x}}(t)$ and of course this is what is specified by Newton's 2nd law. So inorder to ~~speci~~ predict the position of a particle at any later time we must specify at an earlier time the 3 position coordinate and 3 velocity components. These are called the initial conditions. The state of the particle is specified by $\vec{x}$ and $\dot{\vec{x}}$ (at some t), $\{\vec{x}, \dot{\vec{x}}\}$ are called points in configuration space.

So let's consider various initial conditions

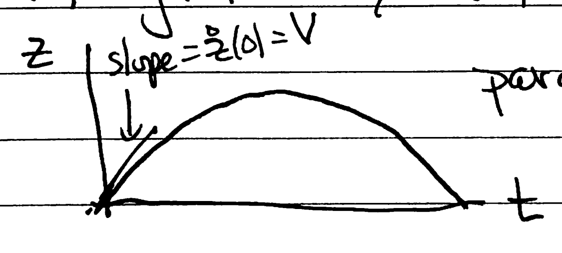
1) Let at $t=0$ $\quad x_0 = y_0 = z_0 = 0$

$$v_y = v_x = 0$$
$$v_z = V$$

i.e. we throw a baseball straight up

$\Rightarrow$

$$x(t) = y(t) = 0 \; ; \; z(t) = Vt - \tfrac{1}{2}gt^2$$



The duration of flight T is the time at which $z(T) = 0$ again

$$z(T) = VT - \tfrac{1}{2}gT^2 = T\left(V - \tfrac{1}{2}gT\right) = 0$$

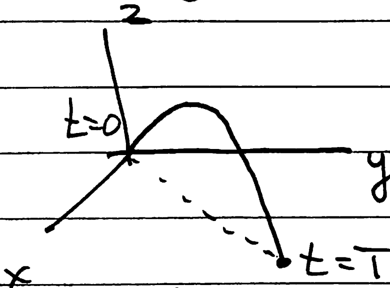
$\rightarrow$ $T = 0$ initial point on ground

$\boxed{T = \frac{2V}{g}}$ duration of flight.

2) General $x_0 = y_0 = z_0 = 0$ but $v_x, v_y, v_z \neq 0$

$\Rightarrow$

$$x = v_x t$$
$$y = v_y t$$
$$z = V_z t - \tfrac{1}{2} g t^2$$



Just uniform motion in x, y directions, with parabola in z. T duration the same ($V \to V_z$)

Example 2: Suppose we consider air resistance or drag and assume that this "frictional" force retards the particle's motion with a magnitude proportional to the speed of the particle. More specifically

$$\vec{F}_{drag} = -b\vec{v} = -b\dot{\vec{x}}$$

(Let's in general use $\vec{r} = \vec{x}$)

The total external force acting on the projectile is gravity + drag (vector sum)

$$\vec{F} = +m\vec{g} - b\vec{v}$$

let $b = km$. The equations of motion are given by Newton's 2nd law

$$m\ddot{\vec{r}} = m\vec{g} - b\dot{\vec{r}} = -mg\hat{k} - mk\dot{\vec{r}}$$

$$\Rightarrow \boxed{\begin{array}{ll} 1) & \ddot{x} + k\dot{x} = 0 \\ 2) & \ddot{y} + k\dot{y} = 0 \\ 3) & \ddot{z} + k\dot{z} + g = 0 \end{array}}$$

We first solve 1 & 2

$$\ddot{x} = -k\dot{x}$$

$$\frac{d}{dt}\dot{x} \quad \Rightarrow \quad \frac{d\dot{x}}{\dot{x}} = -k\,dt$$

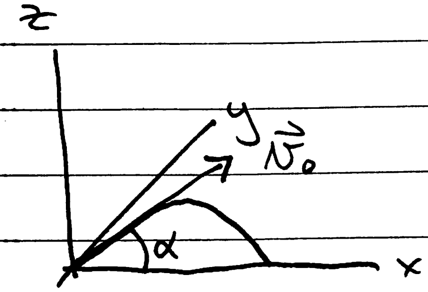
$$\Rightarrow \ln\left(\frac{\dot{x}}{\dot{x}_0}\right) = -k(t-t_0)$$

$$\Rightarrow \boxed{\dot{x}(t) = \dot{x}_0 e^{-k(t-t_0)}}$$

where at $t = t_0$ $\dot{x}(t_0) = \dot{x}_0$
the initial condition for the speed in the x-direction

Similarly $\boxed{\dot{y}(t) = \dot{y}_0 e^{-k(t-t_0)}}$

Wlog we can rotate our coordinate system so that $\dot{y}_0 = 0$ and $\dot{x}_0 = V\cos\alpha$ at t_0; where α is $\measuredangle$ of inclination of projectile's velocity wrt the x-axis



$$\boxed{\begin{aligned} \mathring{x}(t=t_0) &= V\cos\alpha = \mathring{x}_0 \\ \mathring{y}(t=t_0) &= 0 \\ \mathring{z}(t=t_0) &= V\sin\alpha \end{aligned}} \Big\} \text{ i.c.}$$

Similarly, we have chosen the origin of the coordinate system $\Rightarrow$ $\boxed{\vec{r}(t=t_0)=0}$

Now we can integrate the $\mathring{x}(t)$ equation once again

$$\mathring{x}(t) = V\cos\alpha\, e^{-k(t-t_0)}$$

$$\Rightarrow \quad dx = V\cos\alpha\, e^{-k(t-t_0)}\, dt$$

$$\Rightarrow \quad \int_{\substack{x(t_0)=x_0 \\ \| \\ 0}}^{x(t)} dx = V\cos\alpha \int_{t_0}^{t} dt'\, e^{-k(t'-t_0)}$$

$$\Rightarrow \quad x(t) - x(t_0) = V\cos\alpha \left[-\frac{1}{k} e^{-k(t-t_0)} + \frac{1}{k} e^{-k0} \right]$$

$$\boxed{x(t) = x_0 + \frac{V\cos\alpha}{k}\left[1 - e^{-k(t-t_0)}\right]}$$

i.c. $\quad x(t_0) = 0 = x_0$

and $\quad \mathring{x}(t_0) = V\cos\alpha$ ✓

So we find

$$\boxed{x(t) = \frac{V\cos\alpha}{k}\left[1-e^{-k(t-t_0)}\right]}$$

The y-motion is trivial $\dot{y}(t)=0 \Rightarrow y(t)=y_0 = 0$

$$\boxed{y(t)=0}$$

Finally the vertical equation of motion can be integrated twice: first for the z component of $\vec{r}$

$$\frac{d\dot{z}}{dt} = -(g+k\dot{z}) \Rightarrow \frac{d\dot{z}}{g+k\dot{z}} = -dt$$

$$\Rightarrow \int_{\dot{z}(t_0)}^{\dot{z}(t)} \frac{d\dot{z}}{g+k\dot{z}} = -\int_{t_0}^{t} dt' = -(t-t_0)$$

$$\parallel$$

$$\frac{1}{k}\ln(k\dot{z}(t)+g) - \frac{1}{k}\ln(k\dot{z}(t_0)+g)$$

$$\Rightarrow \boxed{k\dot{z}(t)+g = \left[k\dot{z}(t_0)+g\right]e^{-k(t-t_0)}}$$

So

$$\dot{z}(t) = -\frac{g}{k}\left(1-e^{-k(t-t_0)}\right) + \dot{z}(t_0)\,e^{-k(t-t_0)}$$

Now we apply the i.c. $\dot{z}(t_0) = V\sin\alpha$
$\Rightarrow$

$$\boxed{\dot{z}(t) = V\sin\alpha\, e^{-k(t-t_0)} - \frac{g}{k}\left(1-e^{-k(t-t_0)}\right)}$$

Now we can integrate once more

$$z(t) - z(t_0) = \frac{V\sin\alpha}{k}\left(1-e^{-k(t-t_0)}\right) - \frac{g}{k}(t-t_0) + \frac{g}{k^2}\left(1-e^{-k(t-t_0)}\right)$$

Imposing the i.c. $z(t_0) = 0 \Rightarrow$

$$\boxed{z(t) = -\frac{g}{k}(t-t_0) + \frac{kV\sin\alpha + g}{k^2}\left(1-e^{-k(t-t_0)}\right)}$$

Again wlog we can take the initial condition to be at $t_0 = 0$ (shift zero of time)

$\Rightarrow$

$$x(t) = \frac{V\cos\alpha}{k}\left[1 - e^{-kt}\right]$$

$$y(t) = 0$$

$$z(t) = -\frac{gt}{k} + \frac{kV\sin\alpha + g}{k^2}\left[1 - e^{-kt}\right]$$

i.c. $\vec{r}(0) = 0$; $\dot{x}(0) = V\cos\alpha$
$\dot{y}(0) = 0$
$\dot{z}(0) = V\sin\alpha$

As mentioned earlier Newton's 2^{nd} law involves 3 second order DE. Integrating each twice for the trajectory as a function of time $\Rightarrow$ 6 integration constants. Thus we need 6 i.c. in order to prescribe the state of the particle in the future - 3 coordinates
3 speeds.

Rather than plot each coordinate as a function of time we can eliminate the time and solve for a Spatial Trajectory

that is $t = t(x)$ and $z = z(t(x))$

So the $x(t)$ solution $\Rightarrow$

$$\frac{kx}{V\cos\alpha} - 1 = -e^{-kt}$$

$$\Rightarrow \quad -kt = \ln\left[1 - \frac{kx}{V\cos\alpha}\right]$$

hence

$$z = \frac{g}{k^2}\ln\left(1 - \frac{kx}{V\cos\alpha}\right) + \frac{kV\sin\alpha + g}{k^2}\,\frac{kx}{V\cos\alpha}$$

So

$$\boxed{z = \left[\tan\alpha + \frac{g}{kV\cos\alpha}\right]x + \frac{g}{k^2}\ln\left(1 - \frac{kx}{V\cos\alpha}\right)}$$

Now let's study this trajectory in various limits to understand the effects of the different terms.

a.) Suppose $\frac{kx}{V\cos\alpha} \ll 1$ is small then we can expand the logarithm:

$$\ln(1+\epsilon) = \epsilon - \tfrac{1}{2}\epsilon^2 + \tfrac{1}{3}\epsilon^3 + \cdots$$

$$= -\sum_{n=1}^{\infty}(-1)^n\frac{\epsilon^n}{n}$$

This is just found from the Taylor expansion

$$f(\epsilon) = f(0) + \epsilon f'(0) + \frac{\epsilon^2}{2!} f''(0) + \frac{\epsilon^3}{3!} f'''(0) + \cdots$$

$$= \sum_{n=0}^{\infty} \frac{\epsilon^n f^{(n)}(0)}{n!}$$

Now $f(\epsilon) = \ln(1+\epsilon) \Rightarrow f(0) = 0$

$$f'(\epsilon) = \frac{1}{1+\epsilon} \Rightarrow f'(0) = +1$$

$$f''(\epsilon) = -\frac{1}{(1+\epsilon)^2} \Rightarrow f''(0) = -1$$

$$f'''(\epsilon) = \frac{2}{(1+\epsilon)^3} \Rightarrow f'''(0) = +2$$

$$\vdots$$

$$f^{(n)}(\epsilon) = \frac{(-1)^{n-1}(n-1)!}{(1+\epsilon)^n} \Rightarrow f^{(n)}(0) = (-1)^{n-1}(n-1)!$$

$\Rightarrow$

$$\ln(1+\epsilon) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} \epsilon^n \quad \checkmark$$

Thus for $\frac{kx}{V\cos\alpha} \ll 1$ we find

$$Z \simeq \left[\tan\alpha + \cancel{\frac{g}{kV\cos\alpha}}\right] x$$

$$+ \frac{g}{k^2}\left[\cancel{-\frac{kx}{V\cos\alpha}} - \frac{1}{2}\frac{k^2x^2}{V^2\cos^2\alpha} - \frac{1}{3}\frac{k^3x^3}{V^3\cos^3\alpha} + \cdots\right.$$

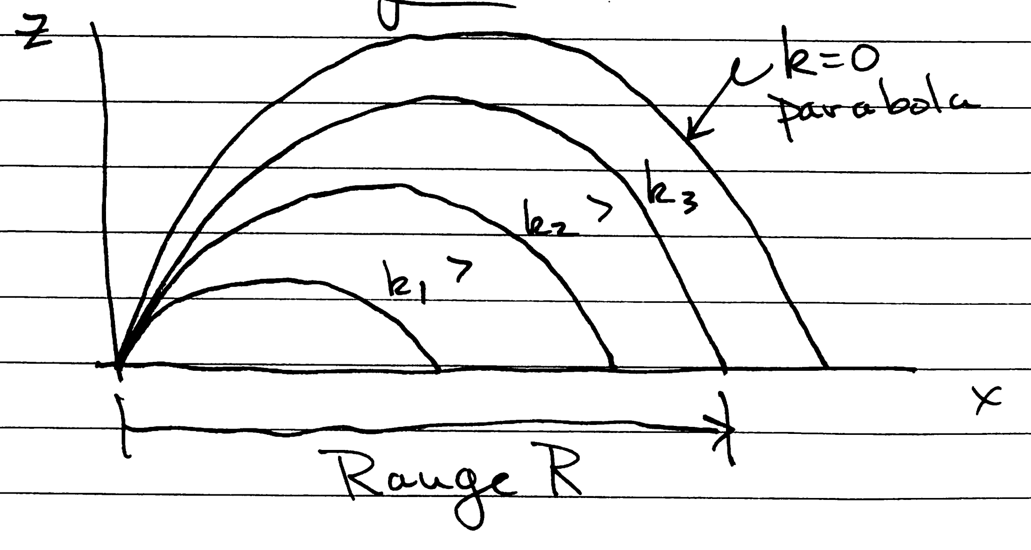
all higher order terms vanish as $k \to 0$

$$Z \simeq x\tan\alpha - \frac{1}{2}\frac{g x^2}{V^2\cos^2\alpha} - \frac{1}{3}\frac{g k x^3}{V^3\cos^3\alpha} + \cdots$$

So as expected for $k \ll 1$ or for small x the projectile starts off with a parabolic trajectory

(same as $k=0$ result) $$Z \simeq x\tan\alpha - \frac{1}{2}\frac{g x^2}{V^2\cos^2\alpha}$$ (indep. of k)

The air resistance force has not had time to build up. As it does build up the trajectory starts to drop off - we have a negative x^3 term



The range R is the value of x where the projectile hits the ground again $Z=0$

For $k=0$ Range $\equiv R_0 = X(Z=0)$

So for $z=0 = R_0 \tan\alpha - \frac{1}{2}\frac{gR_0^2}{V^2\cos^2\alpha}$

$\Rightarrow$

$$0 = \frac{R_0 \frac{1}{2} g}{V^2\cos^2\alpha}\left[-R_0 + \frac{2V^2\cos^2\alpha}{g}\frac{\sin\alpha}{\cos\alpha}\right]$$

$$\Rightarrow R_0 = \frac{V^2}{g}\underbrace{2\sin\alpha\cos\alpha}_{=\sin 2\alpha}$$

$$\boxed{R_0 = \frac{V^2}{g}\sin 2\alpha}$$

Note R_0 is greatest for $\alpha = \frac{\pi}{4} = 45^\circ$.

For small air resistance $k \ll 1$ we can use our approximation for the trajectory to find the range $\left(\frac{kR}{V\cos\alpha} \ll 1\right)$

Thus we will find linear in k correction to R_0

$$0 = R\tan\alpha - \frac{1}{2}\frac{gR^2}{V^2\cos^2\alpha} - \frac{1}{3}\frac{gkR^3}{V^3\cos^3\alpha}$$

$$= R\left[\tan\alpha - \frac{1}{2}\frac{gR}{V^2\cos^2\alpha} - \frac{1}{3}\frac{gkR^2}{V^3\cos^3\alpha}\right]$$

as usual $R=0$ is the initial point we are interested in

$$R = R_0 - \beta k$$

already 1 power of k

$$0 = \left[\cancel{\tan\alpha} - \frac{1}{2}\frac{g}{V^2\cos^2\alpha}(\cancel{R_0} - \beta k) - \frac{1}{3}\frac{gk}{V^3\cos^3\alpha}(R_0 - \beta k)^2\right.$$

ignore these - only first order in k

$\Rightarrow$

$$\frac{1}{2}\frac{g}{V^2\cos^2\alpha}\beta\cancel{k} = \frac{1}{3}\frac{gR_0^2}{V^3\cos^3\alpha}\cancel{k}$$

$$\Rightarrow \quad \boxed{\beta = \frac{2}{3}\frac{R_0^2}{V\cos\alpha}}$$

Thus
$$\boxed{\begin{aligned} R &\simeq R_0\left(1 - \frac{2}{3}\frac{kR_0}{V\cos\alpha}\right) \\ R &\simeq R_0\left(1 - \frac{4}{3}\frac{kV}{g}\sin\alpha\right) \end{aligned}}$$

The change in the Range $\Delta R \equiv R_0 - R$

$$= \frac{4}{3}\frac{kV}{g}\sin\alpha\, R_0$$

$$\boxed{\Delta R = \frac{4}{3}\frac{kV^3}{g^2}\sin\alpha\sin 2\alpha} \geq 0$$

as expected $R < R_0$ the range decreases.

b.) For $\frac{kx}{V\cos\alpha} \simeq 1$ the full trajectory equation must be used to find the Range $z=0$ at $x=R$

$$0 = \left[\tan\alpha + \frac{g}{kV\cos\alpha}\right]R + \frac{g}{k^2}\ln\left(1 - \frac{kR}{V\cos\alpha}\right)$$

This is a transcendental equation which must be solved numerically.

Example 3: Consider the same projectile problem as above but now with the wind having a velocity wrt the earth (frame). Assume the frictional force is proportional to the relative velocity of the projectile to the wind, so

$$m\ddot{\vec{r}} = +m\vec{g} - b(\dot{\vec{r}} - \vec{v}_w) .$$

For constant $\vec{v}_w$, this simply adds a constant force $+b\vec{v}_w$ to $m\vec{g}$ and we can solve as in ex. 2.

Example 4: Effects of altitude

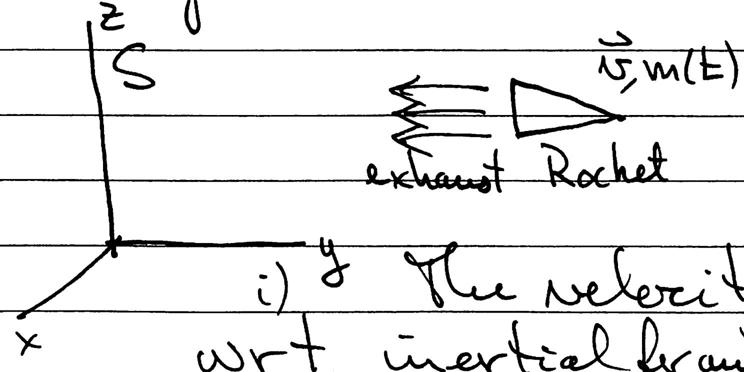
Because the air density decreases exponentially with height, a more accurate equation of motion is

$$m\ddot{\vec{r}} = m\vec{g} - b e^{-\lambda z} \dot{\vec{r}} \qquad \text{with } \lambda \simeq \frac{1}{5\,\text{miles}}$$

This leads to coupled differential equations since z appears in both the x and y equations.

Example 5: Motion With Variable Mass: The Rocket Equation (see R&H p.198)

Our system is the rocket + exhaust material



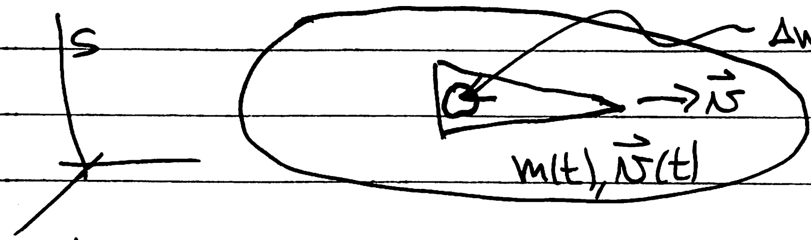
i) The velocity of the rocket wrt inertial frame $S = \vec{v}(t)$

ii) The velocity of exhaust gas relative to the rocket $= \vec{V} =$ constant

iii) The velocity of exhaust gas relative to $S = \vec{V} + \vec{v}(t)$

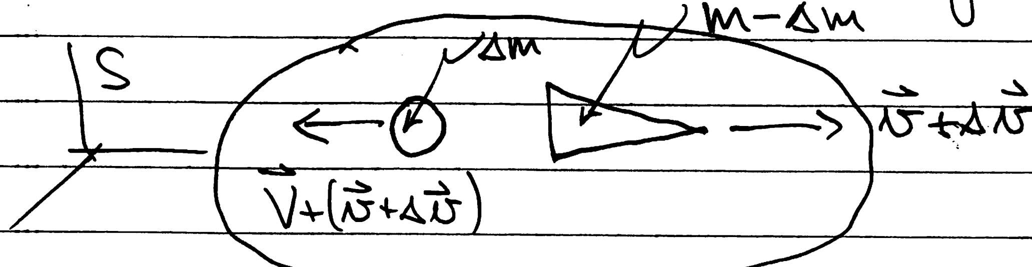
5.) iv) The mass of the rocket + fuel = $m(t)$

So at time t we have the system in state



$\vec{p}(t)$ = momentum of system

At time $t+\Delta t$ we have the system in state



momentum of system = $\vec{p}(t+\Delta t)$

Hence the momentum of the system at t and $t+\Delta t$ is

$$\vec{p}(t) = m(t)\,\vec{v}(t)$$

$$\vec{p}(t+\Delta t) = (m-\Delta m)(\vec{v}+\Delta\vec{v}) + \Delta m\,(\vec{V}+\vec{v}+\Delta\vec{v})$$

So the rate of change of momentum is

$$\frac{\vec{p}(t+\Delta t)-\vec{p}(t)}{\Delta t} = \frac{\cancel{m\vec{v}} + m\Delta\vec{v} - \Delta m\,\vec{v} - \Delta m\Delta\vec{v} + \Delta m\,\vec{V} + \Delta m\,\vec{v} + \Delta m\Delta\vec{v} - \cancel{m\vec{v}}}{\Delta t}$$

$$= m\frac{\Delta\vec{v}}{\Delta t} - \frac{\Delta m}{\Delta t}\vec{v} + \frac{\Delta m}{\Delta t}(\vec{V}+\vec{v})$$

Now the quantity Δm is positive here So that the final mass of the rocket is $m - \Delta m$. It is more convenient to let $\frac{dm}{dt}$ be the actual change (decrease) in the mass of the rocket

$$\frac{dm}{dt} = \frac{m(t+\Delta t) - m(t)}{\Delta t} = \frac{m(t) - \Delta m - m(t)}{\Delta t}$$

$$= -\frac{\Delta m}{\Delta t}$$

We could equally let $\frac{dM_g}{dt}$ be the rate of increase of exhaust gases $\frac{dM_g}{dt} = +\frac{\Delta m}{\Delta t}$, but usually we refer everything to the rocket.

With these conventions in mind

$$\boxed{\frac{d\vec{p}}{dt} = m\dot{\vec{v}} - \dot{m}\vec{V}}$$

$$\left(\text{where } \vec{v}(t+\Delta t) = \vec{v}(t) + \Delta\vec{v} \right.$$
$$\left. \text{so } \dot{\vec{v}} = \frac{\vec{v}(t+\Delta t) - \vec{v}(t)}{\Delta t} = \frac{\Delta\vec{v}}{\Delta t} \right).$$

By Newton's 2nd Law this is just equal to the sum of the external forces acting on the rocket (system)

$$\vec{F} = \frac{d\vec{p}}{dt} = m\dot{\vec{v}} - \dot{m}\vec{V}$$
$$= m\dot{\vec{v}} + \dot{m}\vec{v} - \dot{m}(\vec{V}+\vec{v})$$

$$\boxed{\vec{F} = \frac{d}{dt}(m\vec{v}) - \dot{m}(\vec{V}+\vec{v})}$$

<u>Note</u>: We <u>cannot</u> simply write $\vec{F} = \frac{d}{dt}(m\vec{v})$ for a variable mass system, since this excludes the last term above and does <u>not</u> correspond to a choice of an inertial reference frame in which the 2nd law is valid, but to a special frame in which $\vec{V}+\vec{v}=0$ (exhaust gas frame; not inertial!).

Physically we are most interested in the acceleration of the rocket so

$$m\dot{\vec{v}} = \vec{F} + \underbrace{\dot{m}\vec{V}}_{=\text{ thrust of rocket}}$$

Suppose now we are in free space (no gravity, no friction) where $\vec{F}=0$

$$\Rightarrow \quad m\dot{\vec{v}} = \dot{m}\vec{V}$$

$$\text{or} \quad d\vec{v} = \vec{V}\,\frac{dm}{m}$$

$$\Rightarrow \quad \vec{v}-\vec{v}_0 = \vec{V}\ln\frac{m}{m_0}$$

Now $\vec{v}_0 = \vec{v}(t=0)$ $\quad \vec{v} = \vec{v}(t)$
$m_0 = m(t=0)$ $\quad m = m(t)$ and $m_0 > m(t)$

So $$\boxed{\vec{v}-\vec{v}_0 = -\vec{V}\ln\frac{m_0}{m}}$$

Hence the change in velocity during a given time interval depends only on the exhaust velocity $\vec{V}$ and the total amount of fuel exhausted and not on the rate at which the fuel is exhausted ($\dot{m}$)

Having studied Newton's laws in the specific cases above; let's step back and derive some ~~special cases~~ general consequences of Newton's laws and Properties of Forces.