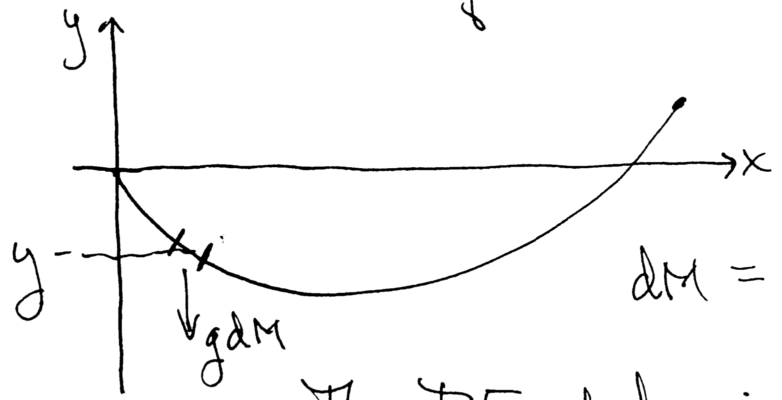
2) The curve is found by requiring the potential energy of the chain to be a minimum (i.e. equilibrium means no net force). Let the zero of Potential Energy be at $y = 0$



$$dM = \rho\, ds \qquad \rho = M/L$$

The PE of ds is $dU = -dM\, g\, y$
$$= -\rho g\, y\, ds$$

Thus
$$\boxed{U[y] = \int_0^a -\rho g\, y\sqrt{1+y'^2}\, dx} \left(= \int_0^a f(y, y'; x)\, dx \right)$$

is the PE of chain and we must find $y = y(x)$ that minimizes it; subject to the constraint
$$\boxed{L = \int_0^a \sqrt{1+y'^2}\, dx} \left(= \int_0^a g(y, y'; x)\, dx \right)$$

Now if we proceed as before

$$\delta U = \int_0^a \left[\frac{\partial f}{\partial y} - \frac{d}{dx}\frac{\partial f}{\partial y'} \right] \alpha \eta(x)\, dx$$

$$= -\rho g \int_0^a \left[\sqrt{1+y'^2} - \frac{d}{dx}\left(\frac{y y'}{\sqrt{1+y'^2}} \right) \right] \alpha \eta(x)\, dx$$

BUT. $\eta(x)$ is no longer arbitrary since

$$L = \int_0^a \sqrt{1+(y'+\alpha\eta')^2}\, dx$$

$$= \int_0^a \sqrt{1+y'^2+2\alpha\eta' y'}\, dx$$

and since $L =$ constant, only those variations that leave $\delta L = 0$ are allowed

$$0 = \delta L = \int_0^a \frac{1}{\sqrt{1+y'^2}} \alpha\eta' y'\, dx$$

$$= -\int_0^a \left(\frac{d}{dx}\left[\frac{y'}{\sqrt{1+y'^2}} \right] \right) \alpha\eta\, dx$$

$$\left(\delta L = \int_0^a \left(\frac{\partial g}{\partial y} - \frac{d}{dx}\frac{\partial g}{\partial y'} \right) \alpha\eta\, dx = 0 \right)$$

The only way for $\delta L = 0$ for $\alpha\eta$ and also $\delta U = 0$ for $\alpha\eta$ is if

$$\left[\frac{\partial f}{\partial y} - \frac{d}{dx}\frac{\partial f}{\partial y'} \right] = \lambda \left[\frac{\partial g}{\partial y} - \frac{d}{dx}\frac{\partial g}{\partial y'} \right]$$

for some constant λ = Lagrange multiplier (independent of x here)

(Crudely put: $\delta L = 0 = \int \left[\frac{d^{\varepsilon} g}{dy}\right] \alpha\, \eta\, dx = 0$
$\eta(x)$ is such that
Then $\delta U = 0$ if $\frac{\partial^{\varepsilon} f}{\partial y} \propto \frac{\partial^{\varepsilon} g}{\partial y}$)

This is equivalent to extremizing

$\int_0^a [f + \lambda g]\, dx$ subject to no constraint but $\eta(x)$ & λ arbitrary but fixed by $L = \text{const} = \int_0^a g\, dx$

Said alternatively: The variation $\delta \int_0^a [f + \lambda g]\, dx$ for arbitrary $\eta(x)$ and λ gives extremization to $U[y] + \lambda L[y]$ with $y = y(x, \lambda)$. We then find the value of $\lambda = \hat{\lambda}$ so that $y = y(x, \hat{\lambda})$ yields $L = \int_0^a g(y, y', x)\, dx$.

This $y(x, \hat{\lambda})$ also extremizes $U + \hat{\lambda} L$ subject to $L[y(x, \hat{\lambda})] = L = \text{constant}$. i.e. under constraints of original problem but $\hat{\lambda} L$ = constant number, hence $U[y]$ is extremized with constraint.

So let's apply this to the above problem

$$\left(\frac{\partial f}{\partial y} - \frac{d}{dx}\frac{\partial f}{\partial y'}\right) = +\lambda\left(\frac{\partial g}{\partial y} - \frac{d}{dx}\frac{\partial g}{\partial y'}\right)$$

Both f & g are independent of x, so multiply by y'

and use

$$\frac{d}{dx}\left[F(y,y') - y'\frac{\partial F}{\partial y'}\right]$$

$$= \left(\frac{\partial F}{\partial y}\frac{dy}{dx} + \cancel{\frac{\partial F}{\partial y'}\frac{dy'}{dx}} - \cancel{y''\frac{\partial F}{\partial y'}} - y'\frac{d}{dx}\frac{\partial F}{\partial y'}\right)$$

$$= \left[\frac{\partial F}{\partial y} - \frac{d}{dx}\frac{\partial F}{\partial y'}\right]y'$$

$\Rightarrow$

$$\left(f - y'\frac{\partial f}{\partial y'}\right) = \lambda\left(g - y'\frac{\partial g}{\partial y'}\right) - k$$

constant

(Recalling: $f = -\rho g\, y\sqrt{1+y'^2}$; $g = \sqrt{1+y'^2}$)

$\Rightarrow$

$$-\rho g\left[y\sqrt{1+y'^2} - \frac{y\,y'^2}{\sqrt{1+y'^2}}\right] = \lambda\left[\sqrt{1+y'^2} - \frac{y'^2}{\sqrt{1+y'^2}}\right] - k$$

Combining terms $\Rightarrow$

$$-\frac{\rho g\, y}{\sqrt{1+y'^2}} = \frac{\lambda}{\sqrt{1+y'^2}} - k$$

or

$$\frac{\rho g\, y + \lambda}{\sqrt{1+y'^2}} = k \quad , \text{ solving for } y' \text{ yields}$$

$$\frac{dy}{dx} = \left[\frac{(\lambda+\rho g y)^2 - k^2}{k^2} \right]^{1/2}$$

So

$$\boxed{\frac{dx}{k} = \frac{dy}{\sqrt{(\lambda+\rho g y)^2 - k^2}}}$$

Let $\lambda + \rho g y = k \cosh\theta$

$$\Rightarrow \quad dy = \frac{k}{\rho g} \sinh\theta \, d\theta$$

& $\sqrt{(\lambda+\rho g y)^2 - k^2} = k\sqrt{\cosh^2\theta - 1} = k \sinh\theta$

$$\Rightarrow \boxed{\frac{dx}{k} = \frac{k}{\rho g} \frac{\sinh\theta \, d\theta}{k \sinh\theta} = \frac{1}{\rho g} d\theta}$$

This implies
$$\frac{x}{k} = \frac{1}{\rho g}\,\theta + \alpha$$
α constant of integration

So
$$\boxed{\theta = \frac{\rho g x}{k} - \alpha}$$

Hence
$$\cosh\theta = \cosh\left[\frac{\rho g x}{k} - \alpha\right]$$

and so
$$(*) \quad \boxed{\lambda + \rho g y = k\cosh\left[\frac{\rho g x}{k} - \alpha\right]}$$

Now λ, k, α are 3 constants determined from

1) $x = 0 \Rightarrow y = 0 \Rightarrow \underline{\lambda = k\cosh\alpha}$

2) $x = a \Rightarrow y = b \Rightarrow \underline{\lambda + \rho g b = k\cosh\left[\frac{\rho g a}{k} - \alpha\right]}$

3) $L = \int_0^a \sqrt{1 + y'^2}\,dx$

Now $\rho g y' = \rho g \sinh\left[\frac{\rho g x}{k} - \alpha\right]$

So that $\sqrt{1 + \sinh^2\left[\frac{\rho g x}{k} - \alpha\right]} = \cosh\left[\frac{\rho g}{k}x - \alpha\right]$

and

$$L = \int_0^a \cosh\left(\frac{\rho g}{k} x - \alpha\right) dx$$

$$= \frac{k}{\rho g} \sinh\left(\frac{\rho g}{k} x - \alpha\right) \Big|_0^a$$

$\Rightarrow$

$$\frac{\rho g L}{k} = \sinh\left(\frac{\rho g a}{k} - \alpha\right) + \sinh\alpha$$

Now 1) $\Rightarrow$ $\cosh\alpha = \lambda/k$ so $\sinh\alpha = \frac{1}{k}\sqrt{\lambda^2 - k^2}$

and we have

$$\frac{\rho g L}{k} - \frac{\sqrt{\lambda^2 - k^2}}{k} = \sinh\left(\frac{\rho g a}{k} - \alpha\right)$$

Summarizing:

1) $\lambda = k\cosh\alpha$

2) $k\cosh\alpha + \rho g b = k\cosh\left(\frac{\rho g a}{k} - \alpha\right)$

3) $\frac{\rho g L}{k} - \sinh\alpha = \sinh\left(\frac{\rho g a}{k} - \alpha\right)$

These can be solved for λ, k, α.

The point being (*) is the equation of the chain — a catenary curve

$$\boxed{y = \frac{k}{\rho g} \cosh\left[\frac{\rho g}{k} x - \alpha\right] - \lambda}$$

These types of constraints are called isoperimetric constraints (i.e. minimum volume when the area is fixed: the perimeter is given by $L = \int \sqrt{1+y'^2}\,dx$ & you extremize the area $A = \int y\,dx$.) Constraints appear as you extremize one functional

$$J[y] = \int_a^b f(y, y'; x)\,dx \quad \text{with}$$

the other functional a constant

$$\text{constant} = L = \int_a^b g(y, y'; x)\,dx \,.$$

The solution is to extremize

isoperimetric constraints)

$J + \lambda L$ (i.e. y & λ independent) with λ an arbitrary constant Lagrange multiplier and $y = y(x, \lambda)$

Then determine λ by requiring

$$L = \int_a^b g(y(x,\lambda), y'(x,\lambda); x)\, dx.$$

Another more common type of constraint involves the coordinates of the path directly in integrated form — these are holonomic constraints

(If the constraints are in differential form, i.e. in terms of velocities not coordinates) these are non-holonomic constraints.) Consider holonomic constraints: Suppose we consider extremizing the functional of several variables

$$J[y_1, \ldots, y_n] = \int_{x_1}^{x_2} f(y_1, \ldots, y_n, y_1', \ldots, y_n'; x)\, dx$$

BUT subject to the constraint that there exists relations between the coordinates

$$g_a(y_1, \dots, y_n; x) = 0, \text{ with } a = 1, \dots, m < n.$$

So we really have a system of $(n-m)$ unknowns to find.

Ex. A particle is constrained to move on the surface of a sphere:

$$x^2 + y^2 + z^2 = \rho^2 = \text{constant}$$

Find the shortest distance of travel between any 2 points by extremizing

$$J = \int_{x_1}^{x_2} \sqrt{1 + y'^2 + z'^2}\, dx \quad \text{subject}$$

to the constraint

$$g(y, z; x) = 0 = x^2 + y^2 + z^2 - \rho^2.$$

Typically we can substitute the constraint directly into the problem — or do this by introducing new (generalized) coordinates so that the constraint is one of the coordinates or at least a simple relation:

In this case introduce spherical polar coordinates so that the constraint is simpler $r=\rho$

$$x=\rho\sin\theta\cos\varphi$$
$$y=\rho\sin\theta\sin\varphi$$
$$z=\rho\cos\theta.$$

The distance interval on the sphere becomes

$$ds^2=dx^2+dy^2+dz^2$$

$$=[\rho\cos\theta\cos\varphi\,d\theta-\rho\sin\theta\sin\varphi\,d\varphi]^2$$
$$+[\rho\cos\theta\sin\varphi\,d\theta+\rho\sin\theta\cos\varphi\,d\varphi]^2$$
$$+[\rho\sin\theta\,d\theta]^2$$

$$=(\rho^2\cos^2\theta\cos^2\varphi\,d\theta^2+\rho^2\sin^2\theta\sin^2\varphi\,d\varphi^2$$
$$\cancel{-2\rho^2 c\theta s\theta c\varphi s\varphi\,d\varphi\,d\theta})$$
$$+(\rho^2c^2\theta s^2\varphi\,d\theta^2+\rho^2s^2\theta\,c^2\varphi\,d\varphi^2$$
$$\cancel{+2\rho^2 s\theta c\theta s\varphi c\varphi\,d\theta\,d\varphi})$$
$$+\rho^2s^2\theta\,d\theta^2$$

$$ds^2=\rho^2(d\theta^2+\sin^2\theta\,d\varphi^2).$$

So the distance becomes:

$$J = \rho \int_{\varphi_1}^{\varphi_2} \sqrt{\left(\frac{d\theta}{d\varphi}\right)^2 + \sin^2\theta}\, d\varphi \quad \left(= \int_{\varphi_1}^{\varphi_2} f(\theta, \theta'; \varphi)\, d\varphi \right)$$

and

$\delta J = 0 \Rightarrow \theta = \theta(\varphi)$ by solving the Euler(-Lagrange) Equation:

$$\frac{\partial f}{\partial \theta} - \frac{d}{d\varphi}\frac{\partial f}{\partial \theta'} = 0$$

with $f = \rho\sqrt{\theta'^2 + \sin^2\theta}$. So multiply $\frac{\partial f}{\partial \theta} - \frac{d}{d\varphi}\frac{\partial f}{\partial \theta'} = 0$ by θ' as earlier to find

$$\frac{d}{d\varphi}\left[f(\theta,\theta') - \theta'\frac{\partial f}{\partial \theta'} \right]$$

$$= \left(\frac{\partial f}{\partial \theta} - \frac{d}{d\varphi}\frac{\partial f}{\partial \theta'} \right)\theta' = 0$$

$\Rightarrow$

$$f(\theta,\theta') - \theta'\frac{\partial f}{\partial \theta'} = \rho a = \text{constant}$$

$$= \rho\sqrt{\theta'^2 + \sin^2\theta} - \frac{\rho\theta'^2}{\sqrt{\theta'^2 + \sin^2\theta}} = a\rho$$

Multiply by f

$$\Rightarrow \quad \rho^2(\cancel{\theta'^2} + \sin^2\theta) - \rho^2\cancel{\theta'^2} = \rho^2 a\sqrt{\theta'^2 + \sin^2\theta}$$

$\Rightarrow$

$$\sin^2\theta = a\sqrt{\theta'^2 + \sin^2\theta}$$

$\Rightarrow$ $$\sin^4\theta = a^2(\theta'^2 + \sin^2\theta)$$

$\Rightarrow$ $$\sin^2\theta[\sin^2\theta - a^2] = a^2\left(\frac{d\theta}{d\varphi}\right)^2$$

$\Rightarrow$ $$\frac{d\varphi}{d\theta} = \frac{a}{\sin\theta\sqrt{\sin^2\theta - a^2}}$$

$$= \frac{a}{\sin^2\theta\sqrt{1 - a^2\csc^2\theta}}$$

So $$\boxed{\frac{d\varphi}{d\theta} = \frac{a\csc^2\theta}{\sqrt{1 - a^2\csc^2\theta}}}$$

Integrate $\Rightarrow$

$$\varphi = \sin^{-1}\left[\frac{\cot\theta}{\beta}\right] + \alpha$$

α is constant of integration

and $\beta^2 = \left(\frac{1-a^2}{a^2}\right)$

So this $\Rightarrow$ $$\boxed{\cot\theta = \beta\sin(\varphi - \alpha)}$$

Multiply by $\rho\sin\theta$

$$\rho \sin\theta \cot\theta = \beta \rho \sin\theta [\sin\varphi \cos\alpha - \cos\varphi \sin\alpha]$$

$\Rightarrow$

$$\rho\cos\theta = (\beta\cos\alpha)\rho\sin\theta\sin\varphi - (\beta\sin\alpha)\rho\sin\theta\cos\varphi$$

$$\parallel$$

$$z = (\beta\cos\alpha)y - (\beta\sin\alpha)x$$

Let $\left.\begin{array}{l} A \equiv \beta\cos\alpha \\ B \equiv \beta\sin\alpha \end{array}\right\}$ both constants

So we finally find

$$\boxed{Ay - Bx = z}$$

This is the equation of a plane passing through the center of the sphere. So the plane's intersection with the surface of the sphere is the curve of least distance between any 2 points on the sphere's surface — it is a great circle. The curve of least distance is called a geodesic. The geodesic on the surface of a sphere is a great circle.

Of course we would like to be able to handle more general problems with holonomic constraints such as a bead sliding along a wire, electromagnetic fields, etc. To do this let's consider the general case again — Now the variation of J is given as before

$$\delta J = \delta\alpha \int_{x_1}^{x_2} \sum_{i=1}^{n} \left(\frac{\partial f}{\partial y_i} - \frac{d}{dx}\frac{\partial f}{\partial y_i'} \right) \eta_i \, dx$$

(let $\alpha \to \delta\alpha$ small)

But the η_i are no longer independent variations of the y_i since

$$g_a(y_1, \ldots, y_n; x) = 0 \quad , \quad a = 1, 2, \ldots, m.$$

So for $\delta\alpha$ small

$$y_i(\alpha, x) = y_i(x) + \delta\alpha\, \eta_i(x)$$

we have also the constraint:

$$g_a(y_1 + \delta\alpha\,\eta_1, y_2 + \delta\alpha\,\eta_2, \ldots, y_n + \delta\alpha\,\eta_n; x) = 0$$

Taylor expanding $\Rightarrow$

$$g_a(y_1,\dots,y_n;x) + \delta\alpha\, \eta_i \frac{\partial g_a}{\partial y_i}(y_1,\dots,y_n;x) = 0$$

But $g_a = 0$, so

$$\sum_{i=1}^{n} \eta_i \frac{\partial g_a}{\partial y_i} = 0 \quad \text{for } a = 1,\dots,m.$$

So we can eliminate the m dependent η_i's

ex.

$$\eta_1 \frac{\partial g_1}{\partial y_1} + \eta_2 \frac{\partial g_1}{\partial y_2} + \dots + \eta_n \frac{\partial g_1}{\partial y_n} = 0$$

$$\vdots \qquad \vdots \qquad \vdots \qquad \vdots$$

$$\eta_1 \frac{\partial g_m}{\partial y_1} + \eta_2 \frac{\partial g_m}{\partial y_2} + \dots + \eta_n \frac{\partial g_m}{\partial y_n} = 0$$

and have in δJ only $(n-m)$ independent variables for example

$n=2$; $m=1$

$$\eta_1 \frac{\partial g_1}{\partial y_1} + \eta_2 \frac{\partial g_1}{\partial y_2} = 0$$

$$\Rightarrow \quad \eta_2 = -\eta_1 \frac{\partial g_1}{\partial y_1} \Big/ \frac{\partial g_1}{\partial y_2}$$

and

$$\delta J = \delta\alpha \int_{x_1}^{x_2} \left[\left(\frac{\partial f}{\partial y_1} - \frac{d}{dx}\frac{\partial f}{\partial y_1'} \right) - \left(\frac{\partial f}{\partial y_2} - \frac{d}{dx}\frac{\partial f}{\partial y_2'} \right) \frac{\partial g_1}{\partial y_1} \Big/ \frac{\partial g_1}{\partial y_2} \right] \eta_1(x)\, dx$$

Now $\eta_1(x)$ is an independent variation. So the only way $\delta J = 0$ is if the integrand is $0 \Rightarrow$

$$\left(\frac{\partial f}{\partial y_1} - \frac{d}{dx}\frac{\partial f}{\partial y_1'} \right)\left(\frac{\partial g_1}{\partial y_1} \right)^{-1} = \left(\frac{\partial f}{\partial y_2} - \frac{d}{dx}\frac{\partial f}{\partial y_2'} \right)\left(\frac{\partial g_1}{\partial y_2} \right)^{-1}$$

$$\equiv -\lambda(x)$$ since each expression is ultimately a function of x; call each side $-\lambda(x)$.

Now this is true for all x, y_1, y_2, thus the LHS and RHS must equal the same function of x independent of y_1, y_2. $\lambda(x)$ is the Lagrange undetermined multiplier.

So we have 3 unknowns now $y_1(x), y_2(x), \lambda(x)$ determined by 3 equations:

1) $$\frac{\partial f}{\partial y_1} - \frac{d}{dx}\frac{\partial f}{\partial y_1'} + \lambda(x)\frac{\partial g_1}{\partial y_1} = 0$$

2) $$\frac{\partial f}{\partial y_2} - \frac{d}{dx}\frac{\partial f}{\partial y_2'} + \lambda(x)\frac{\partial g_1}{\partial y_2} = 0$$

3) $$g_1(y_1, y_2; x) = 0$$

For the general case with y_i; $i = 1, \ldots, n$ and g_a, $a = 1, \ldots, m$ we would have the $(n+m)$ equations for the $(n+m)$ unknowns $(y_1, \ldots, y_n, \lambda_1, \ldots, \lambda_m)$

$$\boxed{\begin{aligned} &\frac{\partial f}{\partial y_i} - \frac{d}{dx}\frac{\partial f}{\partial y_i'} + \sum_{a=1}^{m} \lambda_a(x)\frac{\partial g_a}{\partial y_i} = 0, \quad i = 1, \ldots, n \\ &g_a(y_1, \ldots, y_n; x) = 0, \quad a = 1, \ldots, m. \end{aligned}}$$

So $(n+m)$ unknowns $\{y_i, \lambda_a\}$ and $(n+m)$ equations

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[General Proof: the constraints $\Rightarrow$

$$g_{b,j}\,\eta_j = 0 = g_{b,\hat j}\,\eta_{\hat j} + g_{b,a}\,\eta_a$$

where $\hat j = 1, 2, \dots, n-m$ and $a = n-m+1, \dots, n$ and comma indicates differentiation wrt y_j, etc.

So $\eta_a = -g^{-1}_{ab}\, g_{b,\hat j}\, \eta_{\hat j}$ where $g^{-1}_{ab}\, g_{b,c} = \delta_{ac}$

So δJ becomes

$$\delta J = \int dx \left[(\partial^E_{y_{\hat j}} f)\,\eta_{\hat j} - (\partial^E_{y_a} f)\, g^{-1}_{ab}\, g_{b,\hat j}\, \eta_{\hat j} \right] = 0$$

with $\partial^E_{y_j}$ denoting the Euler derivative wrt y_j

Sine the $\eta_{\hat j}$ are independent $\Rightarrow$

$$\underbrace{\partial^E_{y_{\hat j}} f}_{\substack{\text{function of} \\ y_{\hat j},\, y_{\hat j}' \text{ derivatives} \\ \text{only}}} = \underbrace{(\partial^E_{y_a} f)\, g^{-1}_{ab}}_{\substack{\text{can only be a} \\ \text{function of } x \\ \text{(ultimately} \\ y_i = y_i(x) \\ \xi_a = \xi_a(x)) \\ \text{call it } -\lambda_b(x)}} \; \underbrace{g_{b,\hat j}}_{\substack{\text{function of } y_{\hat j},\, y_{\hat j}' \text{ derivatives} \\ \text{only}}} \quad \text{for each } \hat j$$

So the only way this equality can be satisfied is for

(*) $\partial^{\varepsilon}_{y_i} f = -\lambda_a g_{a,i}$ for all i and $\lambda_a = \lambda_a(x)$

then

$$\partial^{\varepsilon}_{y_a} f \, g^{-1}_{ab} g_{b,\hat{j}} = -\lambda_c g_{c,a} g^{-1}_{ab} g_{b,\hat{j}}$$
$$= -\lambda_c g_{c,\hat{j}}$$

$$\Rightarrow \partial^{\varepsilon}_{y_{\hat{j}}} f = -\lambda_a g_{a,\hat{j}} \quad \checkmark$$

So (*) is

$$\boxed{\begin{array}{l} \dfrac{\partial f}{\partial y_i} - \dfrac{d}{dx}\dfrac{\partial f}{\partial y'_i} + \displaystyle\sum_{a=1}^{m} \lambda_a(x) \dfrac{\partial g_a}{\partial y_i} = 0 \\ \text{and} \quad g_a(y_i; x) = 0 \end{array}}$$

Note that all that was used to implement the constraints was

$$\delta\alpha\, g_{a,i}\,\eta_i = 0$$

& $\delta\alpha\,\eta_i = dy_i$,

So we could also apply our method to solve the differential constraint system $\sum_{i=1}^{n} \frac{\partial g_a}{\partial y_i}\, dy_i = 0$; $a = 1, \dots, m$, also. i.e. $g_a = 0 \Rightarrow \frac{\partial g_a}{\partial y_i}\, dy_i = 0$ so we can ~~treat~~ treat as if holonomic.

An auxiliary variational problem can be introduced in order to obtain the above results from one variational principle. Suppose we treat $\lambda_a = \lambda_a(x)$ just as m-new additional coordinates, then minimize the auxiliary functional

(holonomic constraints)

$$I[y_i, \lambda_a] = J[y_i] + \int_{x_1}^{x_2} \lambda_a(x)\, g_a(y_i; x)\, dx$$

$$= \int_{x_1}^{x_2} \underbrace{\left[f(y_i, y_i'; x) + \sum_{a=1}^{m} \lambda_a(x)\, g_a(y_i; x) \right]}_{\equiv F(y, y', \lambda; x)} dx$$

Now we vary

$$y_i(\alpha, x) = y_i(x) + \alpha \eta_i(x)$$

and

$$\lambda_a(\alpha, x) = \lambda_a(x) + \alpha \epsilon_a(x)$$

with ϵ_a, η_i all treated as independent. This implies

1) $$\frac{\partial F}{\partial y_i} - \frac{d}{dx}\frac{\partial F}{\partial y_i'} = 0$$

2) $$\frac{\partial F}{\partial \lambda_a} - \cancel{\frac{d}{dx}\frac{\partial F}{\partial \lambda_a'}}^{\,0} = 0$$

These become

$$1)\quad \frac{\partial f}{\partial y_i} - \frac{d}{dx}\frac{\partial f}{\partial y_i'} + \frac{\partial g_a}{\partial y_i}\lambda_a = 0$$

$$2)\quad g_a = 0$$

These are just the original constrained problem equations

We have just treated the Lagrange multiplier as an additional independent generalized coordinate.

Finally we can introduce some notation and terminology

Define the <u>variations</u> of J and y by

$$\delta J \equiv [J[y+\delta\alpha\eta] - J[y]]$$
$$= \frac{\partial J}{\partial \alpha}\delta\alpha$$

and

$$\delta y \equiv y(\delta\alpha, x) - y(x) = \delta\alpha\eta = \frac{\partial y}{\partial \alpha}\delta\alpha .$$

The extremum condition is written as <u>the variation of J</u> = 0

$$\delta J = \delta\int_{x_1}^{x_2} f\,dx = 0$$

We treat δ then just as a differential operator

$$\delta J = \int_{x_1}^{x_2} \delta f \, dx \quad \text{with } f = f(y, y'; x) \text{ as usual}$$

fixed limits

$$= \int_{x_1}^{x_2} \left[\frac{\partial f}{\partial y} \delta y + \frac{\partial f}{\partial y'} \delta y' \right] dx$$

but the variation of y' is

$$\delta\left(\frac{dy}{dx}\right) = \delta y' = \frac{d}{dx} y(\delta\alpha, x) - \frac{d}{dx} y(x)$$

$$= \frac{d}{dx}\left[y(\delta\alpha, x) - y(x)\right]$$

$$= \frac{d}{dx}(\delta y) \, .$$

So

$$\delta J = \int_{x_1}^{x_2} \left[\frac{\partial f}{\partial y} \delta y + \frac{\partial f}{\partial y'} \frac{d}{dx} \delta y \right] dx$$

Integrating by parts $\Rightarrow$

$$\delta J = \int_{x_1}^{x_2} \left[\frac{\partial f}{\partial y} - \frac{d}{dx} \frac{\partial f}{\partial y'} \right] \delta y \, dx$$

Since δy is arbitrary, $\delta J = 0 \Rightarrow$

$$\frac{\partial f}{\partial y} - \frac{d}{dx} \frac{\partial f}{\partial y'} = 0 \quad , \text{ as previously.}$$