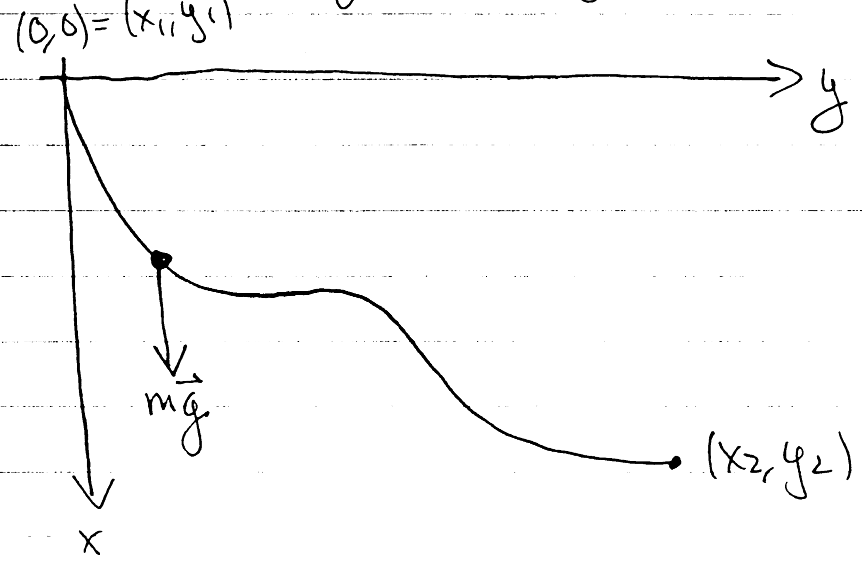
(Chapter 6) Calculus of Variations

Consider the problem of a bead of mass m constrained to slide along a wire curve in a constant gravitational field $\vec{g}$



Suppose we ask the question: What shape should the curved wire be so that the particle initially at rest at the origin arrives at (x_2, y_2) in the least amount of time (this is known as the $\underbrace{\text{brachisto}}_{=\text{"least"}}\underbrace{\text{chrone}}_{=\text{"time"}}$ problem).

Thus we desire to find the equation of the curve: $h(x,y)=0$ or parametrically

$x = x(\alpha)$ where α is some parameter
$y = y(\alpha)$ with $0 \le \alpha \le \alpha_0$ say for $(x_1,y_1) \rightarrow (x_2,y_2)$,

or for that matter $y = y(x)$ describes the curve.

In order to proceed we must relate the time to the position coordinates of the particle. This can be done by using the conservation of energy; that is the total energy of the particle is a constant

$$E = T + U.$$

We choose the zero of potential energy to be at $x=0$; $U(x=0)=0$, so that initially with $v=0$ we have

$$E = U = 0 \text{ at the origin.}$$

In general $U = -mgx$ and $T = \frac{1}{2} m v^2$

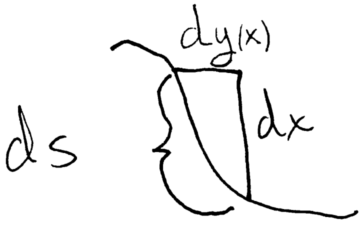
Hence $E = 0 = \frac{1}{2} m v^2 - mgx$

$$\Rightarrow \quad v = \sqrt{2gx} \quad .$$

Now the time required to go from the origin to (x_2, y_2) is just the integral of the distance along the curve (the "arc length") divided by the velocity. More specifically

$$v = \frac{ds}{dt} = \sqrt{2gx}$$

where



ds = distance along curve travelled in time dt

$$= \sqrt{dx^2 + dy^2}$$

$$= \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$$

where $y = y(x)$ describes the curve.

So the time dt to go ds is just

$$dt = \frac{ds(x,y)}{v(x,y)}$$

Hence the total time elapsed is

$$t = \int_{(0,0)}^{(x_2,y_2)} \frac{ds(x,y)}{v(x,y)} = \int_{x=0}^{x_2} \frac{\sqrt{1+\left(\frac{dy}{dx}\right)^2}}{\sqrt{2gx}}\, dx$$

The question we asked is what curve, $y = y(x)$, yields a minimum for the transit time t.

This is a more complicated question than just finding the minimum of an ordinary function, i.e. given a function you only need its value at a point; here the time t depends on the whole functional form of $y(x)$, i.e. many independent functions (∞) each with a different value at, or formerly all points in $[0, x_2]$,

Since the value of the integral depends upon the form of y, i.e. what it is at each x, $y = y(x)$, this t is called a <u>functional</u> of y written

$$t = t[y]$$ (map set of sets into a set, function of ∞ variables).

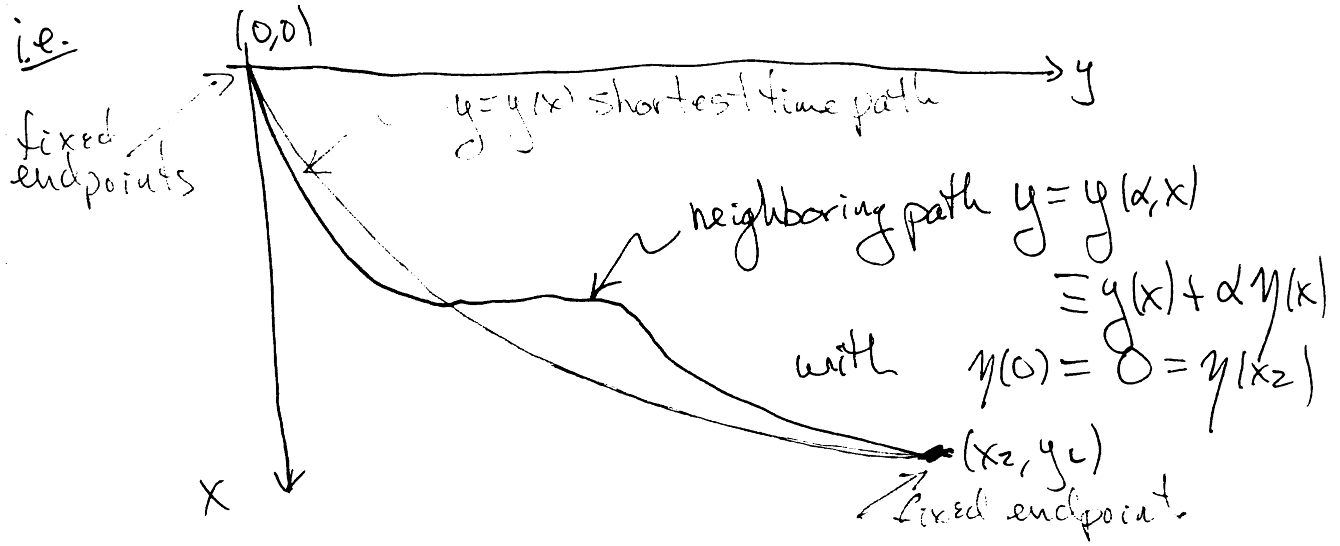
Hence we need to find a condition which will be necessary for extremizing a functional. In order to set up this mathematical requirement in more general terms let $J = J[y]$ be a functional of y where $y = y(x)$ is a function of x, given by

$$J[y] = \int_{x_1}^{x_2} f(y(x), y'(x), x)\, dx$$

with x_1, x_2 fixed and where f is a given function of y, $y' = \frac{dy}{dx}$ and x.

Then J is extremized by y if given any neighboring function, no matter how close to $y(x)$, J of this new function is larger than $J[y]$.

i.e.



then $\boxed{t[y] < t[y+\alpha\eta]}$ for $y(x)$ to be the extremum of t.

Thus we can parameterize our neighboring paths by parameter α (α is positive and negative) by letting

$$y(\alpha,x) \equiv y(x) + \alpha\,\eta(x)$$

where $y(0,\cancel{x}) = y(x)$ and $\eta(x)$ is an arbitrary C^1 function which vanishes at the endpoints of our integral (endpoints are said to be fixed).

So $\eta(x_1) = \eta(x_2) = 0 \Rightarrow \left(y(\alpha,x) = y(x) + \alpha\,\eta(x)\right)$

at $x = x_1$ $\eta(x_1) = 0 \Rightarrow y(\alpha, x_1) = y(x_1)$

$x = x_2$ $\eta(x_2) = 0 \Rightarrow y(\alpha, x_2) = y(x_2)$.

(as we have drawn for our bead path above)

Now let us ask what $J[y+\alpha\eta] > \underset{J[y(0,x)]}{\overset{\shortparallel}{J[y]}}$ implies

So we have

$$J[y(\alpha,x)] > J[y(0,x)].$$

Let's Taylor expand our function about $\alpha = 0$!
(i.e. $J[y(\alpha,x)] = J(\alpha)$)

Recall

$$J[y(\alpha,x)] = J[y+\alpha\eta]$$

$$= \int_{x_1}^{x_2} f\big(y(x)+\alpha\eta(x),\, y'(x)+\alpha\eta'(x);\, x\big)\, dx$$

$$\overset{\text{Taylor expand about } \alpha=0}{=} \int_{x_1}^{x_2} \Big[f(y,y';x) + \alpha\eta(x)\left(\frac{\partial f}{\partial y}\right)\underbrace{(y,y';x)}_{\alpha=0} + \alpha\eta'(x)\left(\frac{\partial f}{\partial y'}\right)\underbrace{(y,y';x)}_{\alpha=0} + O(\alpha^2)\Big] dx$$

$$= J[y] + \alpha \int_{x_1}^{x_2} \left[\frac{\partial f}{\partial y}\eta(x) + \frac{\partial f}{\partial y'}\frac{d\eta}{dx} \right] dx + O(\alpha^2)$$

The last term can be simplified by integrating the $\frac{d\eta}{dx}$ by parts

$$\int_{x_1}^{x_2} \frac{\partial f}{\partial y'} \frac{d\eta}{dx}\, dx = \int_{x_1}^{x_2} \left[\frac{d}{dx}\left(\frac{\partial f}{\partial y'}\eta\right)\right] dx$$

$$- \int_{x_1}^{x_2} \left[\eta(x) \frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right)\right] dx$$

$$= \frac{\partial f}{\partial y'}\eta(x)\Big|_{x=x_1}^{x_2} - \int_{x_1}^{x_2} \eta(x) \frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right) dx$$

But $\eta(x_1) = \eta(x_2) = 0$, the endpoints are fixed

So the first term vanishes. Thus

$$\boxed{J[y+\alpha\eta] = J[y] + \alpha \int_{x_1}^{x_2} \left[\frac{\partial f}{\partial y} - \frac{d}{dx}\frac{\partial f}{\partial y'}\right]\eta(x)\, dx + O(\alpha^2)}$$

For small α the variation in J (differential change in J)

$$\delta J[y] \equiv J[y+\alpha\eta] - J[y].$$

It changes sign when α changes sign, So the only way for $J[y] < J[y+\alpha\eta]$, for all α is for $\delta J = 0$; that is

$$\int_{x_1}^{x_2} \left[\frac{\partial f}{\partial y} - \frac{d}{dx}\frac{\partial f}{\partial y'} \right] \eta(x)\, dx = 0$$

(similarly if $J[y] > J[y+\alpha\eta]$ for all α)

Thus the necessary condition for a maximum or a minimum, i.e. an extremum, of $J\{y]$ is that $\delta J = 0$ (J is stationary)

$$\boxed{\delta J = \int_{x_1}^{x_2} \left[\frac{\partial f}{\partial y} - \frac{d}{dx}\frac{\partial f}{\partial y'} \right] \eta(x)\, dx = 0}$$

But $\eta(x)$ was arbitrary, hence the integrand must vanish $\Rightarrow$

$$\boxed{\frac{\partial f}{\partial y} - \frac{d}{dx}\frac{\partial f}{\partial y'} = 0}$$

This is called the Euler(-Lagrange) equation for f and

$$\delta_y \equiv \frac{\partial}{\partial y} - \frac{d}{dx}\frac{\partial}{\partial y'} \quad \text{is often}$$

called the Euler(-Lagrange) derivative.

To summarize: The functional

$$J[y] = \int_{x_1}^{x_2} f\,(y(x), y'(x); x)\, dx \,,$$

J must be stationary, $\delta J = 0$, and hence

with f given, is extremized by y, then y obeys

$$\delta_y f = 0 = \frac{\partial f}{\partial y} - \frac{d}{dx}\frac{\partial f}{\partial y'} \,;$$

that is if y obeys the Euler-Lagrange equation.

Remarks: 1) This is a necessary condition, for a max. or min., to determine which we need to find the $O(\alpha^2)$ terms for sufficiency.

2) If f is a function of higher than first derivatives, y'', y''', etc., we proceed analogously to find

$$0 = \frac{\partial f}{\partial y} - \frac{d}{dx}\frac{\partial f}{\partial y'} + \frac{d^2}{dx^2}\frac{\partial f}{\partial y''} + \cdots$$

Let's apply these results to our bead on a wire problem: We want to minimize the time of transit from the origin to (x_2, y_2)

$$t\{y\} = \int_{x=0}^{x_2} \frac{\sqrt{1+y'(x)^2}}{\sqrt{2gx}}\, dx$$

where $y'(x) = \frac{dy}{dx}$. The necessary condition for an extremum (one can check it is indeed a minimum) is that

$$\delta t[y] = 0 \Rightarrow \text{Euler-Lagrange equation for } y(x)$$

$$\underbrace{\frac{\partial}{\partial y}\left[\frac{\sqrt{1+y'^2}}{\sqrt{2gx}}\right]}_{\substack{=0 \text{ since} \\ \text{indep. of } y}} - \frac{d}{dx}\frac{\partial}{\partial y'}\left[\frac{\sqrt{1+y'^2}}{\sqrt{2gx}}\right] = 0$$

$$\Rightarrow \quad \frac{d}{dx}\left(\frac{1}{\sqrt{x}}\, y' \frac{1}{\sqrt{1+y'}}\right) = 0$$

$$\Rightarrow \quad \frac{1}{\sqrt{x}} \frac{1}{\sqrt{1+y'}}\, y' = \text{constant in } x$$

squaring

$$\Rightarrow \quad \frac{y'^2}{x(1+y')^2} = \text{constant} \equiv \frac{1}{2a}$$

Solving for y'

$$y'^2 = \frac{x}{2a}(1+y'^2)$$

$$\Rightarrow \quad y'^2\left(1-\frac{x}{2a}\right) = \frac{x}{2a}$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{\sqrt{x}}{(2a-x)^{1/2}} = \frac{x}{(2ax-x^2)^{1/2}}$$

or

$$\boxed{y = \int^x \frac{x\,dx}{\sqrt{2ax-x^2}}}$$

where the constant of integration will be determined later.

Let

$$\boxed{x = a(1-\cos\theta)}$$

$$dx = a\sin\theta\, d\theta$$

$$\left(\begin{aligned} &2ax - x^2 \\ &= 2a^2(1-\cancel{\cos\theta}) \\ &\quad - a^2(1+\cos^2\theta \\ &\qquad - 2\cancel{\cos\theta})) \\ &= a^2(1-\cos^2\theta) = a^2\sin^2\theta \end{aligned}\right.$$

$$y = \int a(1-\cos\theta)\, d\theta$$

$$\Rightarrow$$

$$\boxed{y = a[\theta - \sin\theta] + b}$$

with b = constant of integration

Now at $\theta = 0 \Rightarrow x = 0$ and
So must $y = 0 \Rightarrow \boxed{b = 0}$

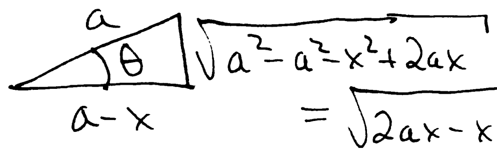
Hence we find the parametric equation for the shortest time curve

$$\boxed{\begin{aligned} x &= a(1-\cos\theta) \\ y &= a(\theta - \sin\theta) \end{aligned}}$$

This is called a cycloid curve. It passes through the origin and a is determined so that $y = y_2$ at $x = x_2$.

i.e.

$$\cos\theta = \frac{a-x}{a}$$



$$= \sqrt{2ax-x^2}$$

$$\theta = \cos^{-1}\frac{a-x}{a}$$
$$= \sin^{-1}\frac{\sqrt{2ax-x^2}}{a}$$

So

$$y = a\left[\sin^{-1}\frac{\sqrt{2ax-x^2}}{a} - \frac{\sqrt{2ax-x^2}}{a}\right]$$

hence we have $y = y(x)$ and at $x = x_2$; $y = y_2$

$$y_2 = a\left[\sin^{-1}\frac{\sqrt{2ax_2-x_2^2}}{a} - \frac{\sqrt{2ax_2-x_2^2}}{a}\right]$$

This is an equation for $a = a(x_2, y_2)$.

So recall for an ordinary function of a single variable $f(x)$

The x that minimizes ~~of~~ the function is $\ni$ (denote x_0)

$$f(x_0) < f(x_0 + \alpha dx)$$

$$f(x_0+\alpha dx) = f(x_0) + \alpha dx \left.\frac{\partial f}{\partial x}\right|_{x=x_0}$$

So

$$df = f(x_0+\alpha dx) - f(x_0) = \alpha dx \left.\frac{\partial f}{\partial x}\right|_{x=x_0}$$

(i.e. $dx = \pm$)

Since α is $\pm$ it is necessary for $df = 0$ (but not sufficient) for a minimum — it extremizes f $\Rightarrow$ $\left.\frac{\partial f}{\partial x}\right|_{x=x_0} = 0$ $\Rightarrow$ equation for x_0

Analogously

$$J[y] = \int_{x_1}^{x_2} f(y, \tfrac{dy}{dx}; x)\, dx$$ is a functional of y

$y(x)$ minimizes J if $J[y] < J[y + \alpha\eta(x)]$

So Taylor expand about $\alpha = 0$

$$J[y+\alpha\eta] = \cancel{J[y] + \int_{x_1}^{x_2} dx}$$

$$= \int_{x_1}^{x_2} dx \left[f(y, y'; x) + \alpha \left.\frac{\partial f}{\partial \alpha}\right|_{\alpha=0} \right]$$

$$\left.\frac{\partial f}{\partial y}\right|_{\alpha=0} \frac{\partial(\alpha\eta)}{\partial \alpha} + \left.\frac{\partial f}{\partial y'}\right|_{\alpha=0} \frac{\partial \alpha\eta'}{\partial \alpha}$$

$$J[y+\alpha\eta] = J[y] + \alpha\int_{x_1}^{x_2} dx \left[\eta \left.\frac{\partial f}{\partial y}\right|_{\alpha=0} + \frac{d\eta}{dx}\left.\frac{\partial f}{\partial y'}\right|_{\alpha=0} \right]$$

$$= J[y] + \alpha\int_{x_1}^{x_2} dx\, \eta(x)\left[\frac{\partial f}{\partial y}(y,y';x) - \frac{d}{dx}\frac{\partial f}{\partial y'}(y,y';x)\right]$$

$$+ \alpha\left[\underbrace{\eta(x_2)}_{\to 0}\frac{\partial f}{\partial y'}(y(x_2),y'(x_2);x_2) - \underbrace{\eta(x_1)}_{\to 0}\frac{\partial f}{\partial y'}(y(x_1),y'(x_1);x_1)\right]$$

fixed endpts.

So

$$\delta J = J[y+\alpha\eta] - J[y] = \alpha\int_{x_1}^{x_2} dx\, \eta(x)\left[\frac{\partial f}{\partial y}(y,y';x) - \frac{d}{dx}\frac{\partial f}{\partial y'}(y,y';x)\right]$$

$\alpha = \pm$ So nec. condition for minimum is $\delta J = 0$

Since $\eta(x)$ is arb. $\Rightarrow$ Euler-Lagrange eq. for f

$$\boxed{\frac{\partial f}{\partial y} - \frac{d}{dx}\frac{\partial f}{\partial y'} = 0 \quad \text{This is a diff. Eq. for } y(x).}$$

The brachistochrone problem is particularly simple. In general we will have integrals which depend on more than one function of a real variable to extremize; that is we will consider <u>functionals of several variables</u>

$$J[y_1, y_2, \ldots, y_n] = \int_{x_1}^{x_2} f(y_1, \ldots, y_n, y_1', \ldots, y_n'; x)\,dx$$

where $y_i = y_i(x)$. We desire the form of the functions $y_i = y_i(x)$ which extremize J. Towards this end consider paths in this multi-dimensional space

$$y_i(\alpha, x) \equiv y_i(x) + \alpha\,\eta_i(x) \quad \text{with}$$

η_i independent C' functions $\ni \eta_i(x_1) = \eta_i(x_2) = 0.$

Then as before

$$\delta J = \alpha \sum_{i=1}^{n} \int_{x_1}^{x_2} \left[\frac{\partial f}{\partial y_i} - \frac{d}{dx}\frac{\partial f}{\partial y_i'} \right] \eta_i(x)\,dx$$

in or out

If J is extremized by $\{y_i\}$, then $\delta J = 0$

$$\Rightarrow \quad \frac{\partial f}{\partial y_i} - \frac{d}{dx}\frac{\partial f}{\partial y_i'} = 0 \quad \text{for each } i = 1, \ldots, n$$

since the η_i are independent variations of the functions.

Even this "n-dimensional" functional problem is simple. In many cases we desire the shortest path between two points when the path is additionally constrained. For example a particle moving on the surface of a sphere, then $x^2+y^2+z^2=R^2$, x, y, z are not independent variables.

Let's consider the case of ordinary functions of 2-variables ($F = F(x,y)$).

If $F(x,y)$ is stationary at (x_0, y_0) then

$$dF = 0 = \frac{\partial F}{\partial x}dx + \frac{\partial F}{\partial y}dy \quad \text{at } (x_0, y_0)$$

Since dx & dy are indep.

F is stationary ($dF=0$) iff $\frac{\partial F}{\partial x} = 0 = \frac{\partial F}{\partial y}$ at (x_0, y_0)

Now suppose that x & y are not indep. but there is a constraint among (x, y)

$$G(x,y) = 0 .$$

$$\Rightarrow \quad dG = 0 = \frac{\partial G}{\partial x}dx + \frac{\partial G}{\partial y}dy$$

Suppose $\frac{\partial G}{\partial y} \neq 0$ then

$$dy = -\frac{\partial G/\partial x}{\partial G/\partial y}dx \quad \text{at } (x_0, y_0)$$

So

$$dF = \left[\frac{\partial F}{\partial x}\frac{\partial G}{\partial y} - \frac{\partial G}{\partial x}\frac{\partial F}{\partial y}\right]\frac{dx}{\partial G/\partial y}$$

For F to be stationary a nec. condition is, since dx is indep. variable

$$dF = 0 \Rightarrow$$

$$\frac{\partial F}{\partial x}\frac{\partial G}{\partial y} - \frac{\partial G}{\partial x}\frac{\partial F}{\partial y} = 0$$

$\Rightarrow$ (i.e. $\frac{\partial F}{\partial x}\left(\frac{\partial G}{\partial x}\right)^{-1} = \frac{\partial F}{\partial y}\left(\frac{\partial G}{\partial y}\right)^{-1}$ at (x_0, y_0). The LHS = RHS = a number call it λ $= \lambda$)

~~This is true if~~ This is true if

$$\frac{\partial F}{\partial x} = -\lambda\frac{\partial G}{\partial x}$$

and

$$\frac{\partial F}{\partial y} = -\lambda\frac{\partial G}{\partial y}$$

Alternative said:

1) $dF = 0$, $dG = 0$ $\Rightarrow$ $\begin{pmatrix} F_x & F_y \\ G_x & G_y \end{pmatrix} \begin{pmatrix} dx \\ dy \end{pmatrix} = 0$

for $\begin{matrix} dx \neq 0 \\ dy \neq 0 \end{matrix} \Rightarrow \det \begin{vmatrix} F_x & F_y \\ G_x & G_y \end{vmatrix} = 0$

$\Rightarrow \begin{matrix} F_x = -\lambda G_x \\ F_y = -\lambda G_y \end{matrix}$ 1st row is a multiple of 2nd row.

2) Geometrically $\begin{matrix} dF = 0 \Rightarrow dF = \vec{\nabla} F \cdot d\vec{r} = 0 \\ dG = 0 \Rightarrow dG = \vec{\nabla} G \cdot d\vec{r} = 0 \end{matrix}$

$\Rightarrow \begin{matrix} \vec{\nabla} F \perp d\vec{r} \\ \vec{\nabla} G \perp d\vec{r} \end{matrix} \Rightarrow \vec{\nabla} F \parallel \vec{\nabla} G$

$\Rightarrow \begin{matrix} F_x = -\lambda G_x \\ F_y = -\lambda G_y \end{matrix}$

~~So we have an equation for λ~~

$\cancel{\lambda = -\dfrac{\partial F/\partial y}{\partial G/\partial y}}$

So we have our necessary condition

$F_x = -\lambda G_x$

$F_y = -\lambda G_y$ where $\lambda = -F_y/G_y \Big|_{(x_0, y_0)}$ (unique λ)

~~This is~~ The combination of conditions

If $\frac{\partial F}{\partial x} + \lambda \frac{\partial G}{\partial x} = 0$ and $\frac{\partial F}{\partial y} + \lambda \frac{\partial G}{\partial y} = 0$ at (x_0, y_0)

~~then~~ with a constraint at (x_0, y_0) $G(x_0, y_0) = 0$ $(= c)$

$\Rightarrow$ $\frac{dG}{dx} dx + \frac{dG}{dy} dy = 0$ So dx & dy are consistent with the constraint

So then

$$dx\left[\frac{\partial F}{\partial x} + \lambda \frac{\partial G}{\partial x}\right] = 0$$

$$+ \; dy\left[\frac{\partial F}{\partial y} + \lambda \frac{\partial G}{\partial y}\right] = 0$$

$\Rightarrow$ $\frac{\partial F}{\partial x} dx + \frac{\partial F}{\partial y} dy = 0 \Rightarrow dF = 0$ at (x_0, y_0)

This is sufficient condition for stationarity of F at (x_0, y_0) subject to constraint $G = 0$

So we have recast the constraint λ problem to an equivalent problem

F is stationary at (x_0, y_0) with (dx, dy) subject to the constraint that $G(x, y) = 0$

iff $F + \lambda G$ is stationary at (x_0, y_0)

for some (unique) λ for arbitrary (dx, dy) and $d\lambda$.

That is

$$d(F+\lambda G)=0\left(=\frac{\partial H}{\partial x}dx+\frac{\partial H}{\partial y}dy+\frac{\partial H}{\partial \lambda}d\lambda\right)$$

all indep. now.

$$\Rightarrow \quad \frac{\partial F}{\partial x}+\lambda\frac{\partial G}{\partial x}=0$$

$$\frac{\partial F}{\partial y}+\lambda\frac{\partial G}{\partial y}=0$$

AND $\quad \frac{\partial}{\partial\lambda}(F+\lambda G)=0 \Rightarrow G=0$

So it is like 3 coord. (x, y, λ) with no constraint BUT ~~a~~ new Function

$$H=F+\lambda G \quad \text{to extremize}$$

in terms of (x, y, λ).

Now consider the example

1) $F(x,y)=xy$ and $G(x,y)=x^2+y^2-1=0$

a.) $y=\pm\sqrt{1-x^2} \Rightarrow F=\pm x\sqrt{1-x^2}$

$$\frac{dF}{dx}=\frac{\pm(1-2x^2)}{\sqrt{1-x^2}}$$

$$=0 \Rightarrow x=\pm\frac{1}{\sqrt{2}}$$

$$\Rightarrow y=\pm\frac{1}{\sqrt{2}}$$

So $dF=0$ at $\left.\begin{array}{l}x=\pm\frac{1}{\sqrt{2}}\\ y=\pm\frac{1}{\sqrt{2}}\end{array}\right\}$ 4 points

$\left(\frac{1}{\sqrt{2}},\frac{1}{\sqrt{2}}\right),\left(\frac{-1}{\sqrt{2}},\frac{-1}{\sqrt{2}}\right) \Rightarrow F=1/2$

$\left(\frac{1}{\sqrt{2}},\frac{-1}{\sqrt{2}}\right),\left(\frac{-1}{\sqrt{2}},\frac{1}{\sqrt{2}}\right) \Rightarrow F=-\frac{1}{2}$

1.b) Equivalent method: Lagrange Multiplier λ

$$H = F + \lambda G$$

1) $\frac{\partial H}{\partial x} = y + 2\lambda x = 0$

2) $\frac{\partial H}{\partial y} = x + 2\lambda y = 0 \Rightarrow \lambda = -\frac{x}{2y} \quad (y \neq 0)$

3) $\frac{\partial H}{\partial \lambda} = G = x^2 + y^2 - 1 = 0$

2) into 1) $\Rightarrow \quad y - 2\frac{x}{2y}x = 0 \Rightarrow y^2 - x^2 = 0 \Rightarrow \boxed{x^2 = y^2}$

3) $\Rightarrow \quad x^2 = 1 - x^2 \Rightarrow 2x^2 = 1 \Rightarrow \boxed{x = \pm\frac{1}{\sqrt{2}}}$

$\Rightarrow \boxed{y = \pm\frac{1}{\sqrt{2}}}$ and $\boxed{\lambda = \pm 1/2}$

x	y	λ	F
$\frac{1}{\sqrt{2}}$	$\frac{1}{\sqrt{2}}$	$-\frac{1}{2}$	$1/2$
$\frac{1}{\sqrt{2}}$	$-\frac{1}{\sqrt{2}}$	$+\frac{1}{2}$	$-1/2$
$-\frac{1}{\sqrt{2}}$	$\frac{1}{\sqrt{2}}$	$+\frac{1}{2}$	$-1/2$
$-\frac{1}{\sqrt{2}}$	$-\frac{1}{\sqrt{2}}$	$-\frac{1}{2}$	$+1/2$

(or 1) $\Rightarrow xy + 2\lambda x^2 = 0$

2) $\Rightarrow xy + 2\lambda y^2 = 0$

$(x \neq 0,\ y \neq 0$ not stationary pt.)

$2xy + 2\lambda = 0$

$\boxed{\lambda = -xy}$

1) $\Rightarrow y(1 - 2x^2) = 0$

3) $\Rightarrow x^2 + y^2 = 1$)

The Lagrange multiplier technique works in more general case

2) Suppose $F(x,y,z) = xyz$ & $G = x^2+y^2+z^2-1 = 0$

Then $H = F + \lambda G$

1) $\frac{\partial H}{\partial x} = yz + 2\lambda x = 0 \Rightarrow xyz + 2\lambda x^2 = 0$

2) $\frac{\partial H}{\partial y} = xz + 2\lambda y = 0 \Rightarrow xyz + 2\lambda y^2 = 0$

3) $\frac{\partial H}{\partial z} = xy + 2\lambda z = 0 \Rightarrow xyz + 2\lambda z^2 = 0$

4) $\frac{\partial H}{\partial \lambda} = G = x^2+y^2+z^2-1 = 0$

Add 1,2,3 $\Rightarrow$ $3xyz + 2\lambda\underbrace{(x^2+y^2+z^2)}_{4) \Rightarrow = 1} = 0$

$\Rightarrow$ $\boxed{\lambda = -\frac{1}{2}(3xyz)}$

1) $\Rightarrow$ $xyz(1-3x^2) = 0$

2) $\Rightarrow$ $xyz(1-3y^2) = 0$

4) $\Rightarrow$ $x^2+y^2+z^2 = 1$

(3) $\Rightarrow$ $xyz(1-3z^2) = 0 = xyz[-2+3x^2+3y^2]$
this is just sum of 1 & 2 nothing new)

Cases: $x = \pm 1,\ y = 0,\ z = 0$
$x = 0,\ y = \pm 1,\ z = 0$
$x = 0,\ y = 0,\ z = \pm 1$
$x = \pm \frac{1}{\sqrt{3}},\ y = \pm \frac{1}{\sqrt{3}},\ z = \pm \frac{1}{\sqrt{3}}$

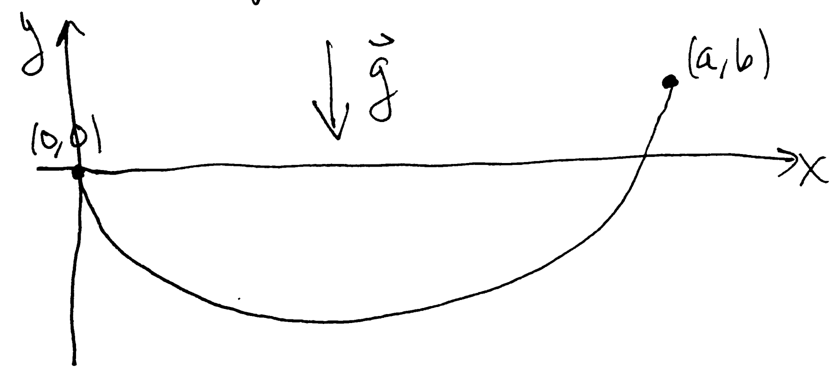
part of circle below

circles
$z = 0 \Rightarrow x^2 + y^2 = 1$ circle
$x = 0 \Rightarrow y^2 + z^2 = 1$ "
$y = 0 \Rightarrow x^2 + z^2 = 1$

$x = \pm \frac{1}{\sqrt{3}},\ y = 0,\ z = \pm \sqrt{\frac{2}{3}}$
etc.

A simpler warm up constraint problem involves an overall constraint

ex A heavy chain of fixed length L is hung between the points $(0,0)$ and (a,b) as shown



It has uniform mass density ρ and total mass M and is subject to a constant gravitational force in the $(-y)$-direction.

What will be its equilibrium shape?
(The curve is called a catenary)

1) First we have the constraint that the length of the chain is L. If $y = y(x)$, then the arc length along the curve is

$$ds = \sqrt{dx^2+dy^2} = \sqrt{1+y'^2}\,dx$$

So

$$L = \int_0^a \sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx \quad \left(= \int_0^a g(y,y';x)\,dx\right)$$