4 Harmonic Oscillator

As we mentioned initially the utility of analyzing the (harmonic) sinusoidal force problem comes from being able to decompose any periodic force in terms of sums of sines and cosines a lá Fourier series. More precisely we can use the principle of superposition which our linear differential operator obeys.
Theorem: Let the set of functions (finite or infinite) $x_n(t)$, with $n = 1, 2, \ldots$ be solutions of the equations

$$Lx_n(t) = F_n(t), \tag{4.1}$$

where L is a linear operator

$$L(\alpha x_1 + x_2) = \alpha L x_1 + L x_2, \tag{4.2}$$

for example

$$L = m\frac{d^2}{dt^2} + b\frac{d}{dt} + k, \tag{4.3}$$

then if

$$F(t) = \sum_n F_n(t), \tag{4.4}$$

the solution of

$$Lx(t) = F(t) \tag{4.5}$$

is

$$x(t) = \sum_n x_n(t). \tag{4.6}$$

Proof:

$$Lx(t) = L\sum_n x_n(t) \overset{\text{by linearity of } L}{\overbrace{=}} \sum_n Lx_n(t) = \sum_n F_n = F(t). \tag{4.7}$$

Now for the driven harmonic oscillator, suppose we consider a driving force of the form

$$F(t) = \sum_n F_n(t) = \sum_n \alpha_n \cos(\omega_n t - \phi_n). \tag{4.8}$$

We desire a solution of

$$m\ddot{x} + b\dot{x} + kx = F(t). \tag{4.9}$$

Then by the above theorem the particular solution x_p is

$$x_p(t) = \sum_n x_n(t), \tag{4.10}$$

where x_n is a solution of

$$m\ddot{x}_n + b\dot{x}_n + kx_n = \alpha_n \cos(\omega_n t - \phi_n). \tag{4.11}$$

Now we already know the particular solution of this equation, it is given by

$$x_n(t) = \frac{\alpha_n}{m} \frac{\cos(\omega_n t - \phi_n)}{\sqrt{(\omega_0^2 - \omega_n^2)^2 + 4\omega_n^2\beta^2}}, \tag{4.12}$$

where

$$\tan\delta_n = \left(\frac{2\omega_n\beta}{\omega_0^2 - \omega_n^2}\right). \tag{4.13}$$

Hence the particular solution for $F(t)$ is given by the superposition of the x_n solutions

$$x_p(t) = \sum_n x_n(t) = \sum_n \frac{\alpha_n}{m} \frac{\cos(\omega_n t - \phi_n)}{\sqrt{(\omega_0^2 - \omega_n^2)^2 + 4\omega_n^2\beta^2}}, \tag{4.14}$$

while the most general solution to equation (4.9) includes the solution of the homogeneous equation also

$$x(t) = Ae^{-\beta t}\cos(\omega_h t + \theta) + x_p(t), \tag{4.15}$$

where A and θ in the transient (homogeneous) solution are determined by the initial conditions and $\omega_h \equiv \sqrt{\omega_0^2 - \beta^2}$.

Thus whenever the force can be decomposed into a sum (finite or infinite) of cosine terms we can find the solution of (4.9) by superposition of solutions for each term in the sum. We can also do the same if the force is in terms of the sine function. We can either return to our complex equation and take the imaginary part yielding

$$m\ddot{x} + b\dot{x} + kx = F_0 \sin(\omega t + \theta_0) \tag{4.16}$$

with particular solution

$$\begin{aligned} x_p(t) &= \mathrm{Im}X = \frac{F_0}{m} \frac{[(\omega_0^2 - \omega^2)\sin(\omega t + \theta_0) - 2\beta\omega\cos(\omega t + \theta_0)]}{(\omega_0^2 - \omega^2)^2 + 4\beta^2\omega^2} \\ &= \frac{F_0}{m} \frac{[\cos\delta \sin(\omega t + \theta_0) - \sin\delta\cos(\omega t + \theta_0)]}{\sqrt{(\omega_0^2 - \omega^2)^2 + 4\beta^2\omega^2}} \\ x_p(t) &= \frac{F_0}{m} \frac{1}{\sqrt{(\omega_0^2 - \omega^2)^2 + 4\beta^2\omega^2}} \sin(\omega t + \theta_0 - \delta). \end{aligned} \tag{4.17}$$

Formally for $F_n = \alpha_n \cos(\omega_n t - \phi_n)$ just let $\phi_n \to \phi_n + \pi/2$, then $\cos(\omega_n t - \phi_n) \to \sin(\omega_n t - \phi_n)$ and $\cos(\omega_n t - \phi_n - \delta_n) \to \sin(\omega_n t - \phi_n - \delta_n)$.

Thus for

$$F(t) = \sum_n \beta_n \sin(\omega_n t - \phi_n) \tag{4.18}$$

the solution of equation (4.9) is given by

$$x_p(t) = \sum_n x_n(t), \tag{4.19}$$

with x_n the particular solution of

$$m\ddot{x}_n + b\dot{x}_n + kx_n = \beta_n \sin(\omega_n t - \phi_n). \tag{4.20}$$

This is

$$x_n(t) = \frac{\beta_n}{m} \frac{\sin(\omega_n t - \phi_n - \delta_n)}{\sqrt{(\omega_0^2 - \omega_n^2)^2 + 4\beta^2\omega_n^2}}, \tag{4.21}$$

where

$$\tan\delta_n = \left(\frac{2\omega_n\beta}{\omega_0^2 - \omega_n^2}\right). \tag{4.22}$$

Then the most general solution has the form

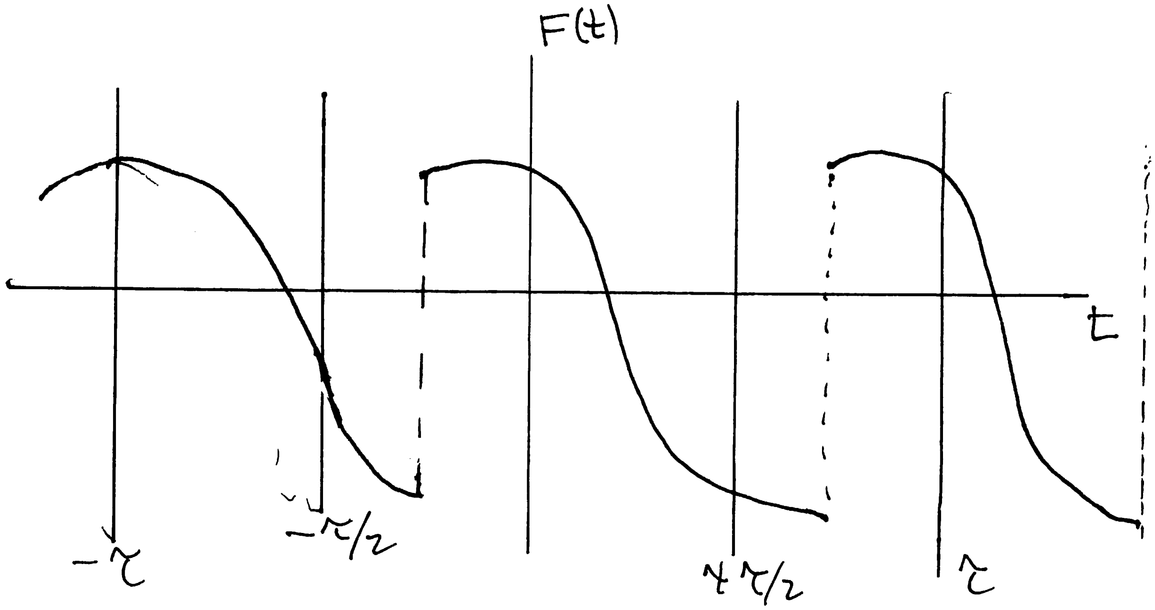
$$x(t) = Ae^{-\beta t}\cos(\omega_h t + \theta) + x_p(t). \tag{4.23}$$

Now the power of these results is that any arbitrary periodic force function (piecewise continuous) can be represented according to Fourier's Theorem by a series of harmonic terms–a Fourier series. Thus if $F(t)$ is periodic with period $\tau = 2\pi/\omega$, $F(t+\tau) = F(t)$, then by Fourier's Theorem we have that

$$F(t) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty}\left[a_n\cos(n\omega t) + b_n\sin(n\omega t)\right], \tag{4.24}$$

with $\omega 2\pi/\tau$ and where the coefficients a_n and b_n are real numbers given by

$$\begin{aligned}
a_n &= \frac{2}{\tau}\int_0^{\tau} dtF(t)\cos(n\omega t) \\
&= \frac{\omega}{\pi}\int_{-\frac{\pi}{\omega}}^{+\frac{\pi}{\omega}} dtF(t)\cos(n\omega t) \\
b_n &= \frac{2}{\tau}\int_0^{\tau} dtF(t)\sin(n\omega t) \\
&= \frac{\omega}{\pi}\int_{-\frac{\pi}{\omega}}^{+\frac{\pi}{\omega}} dtF(t)\sin(n\omega t).
\end{aligned} \tag{4.25}$$



To summarize: the most general solution to the driven harmonic oscillator equation of motion

$$m\ddot{x} + b\dot{x} + kx = F(t) \tag{4.26}$$

is given by

$$x(t) = Ae^{-\beta t}\cos(\omega_h t + \theta) + x_p(t) \tag{4.27}$$

with

$$x_p(t) = +\frac{a_0}{m}\frac{1}{2\omega_0} + \sum_{n=1}^{\infty}\frac{\frac{a_n}{m}\cos(\omega_n t - \delta_n) + \frac{b_n}{m}\sin(\omega_n t - \delta_n)}{\sqrt{(\omega_0^2 - \omega_n^2)^2 + 4\beta^2\omega_n^2}}, \tag{4.28}$$

where $\omega_n \equiv n\omega = 2\pi n/\tau$, with $n = 1, 2, \ldots$, and

$$\tan\delta_n = \left(\frac{2\omega_n\beta}{\omega_0^2 - \omega_n^2}\right). \tag{4.29}$$

Now let's step back and rewrite the Fourier series for the force, equation (4.24), by recalling

$$\begin{aligned} \sin x &= \frac{e^{ix} - e^{-ix}}{2i} \\ \cos x &= \frac{e^{ix} + e^{-ix}}{2}. \end{aligned} \tag{4.30}$$

Hence

$$F(t) = \frac{1}{2}a_0 e^0 + \sum_{n=1}^{\infty}\left[\left(\frac{a_n}{2} + \frac{b_n}{2i}\right)e^{in\omega t} + \left(\frac{a_n}{2} - \frac{b_n}{2i}\right)e^{-in\omega t}\right]$$

$$= \frac{1}{2}a_0 e^0 + \sum_{n=1}^{\infty}\left[\frac{1}{2}(a_n - ib_n)\, e^{in\omega t} + \frac{1}{2}(a_n + b_n)\, e^{-in\omega t}\right]. \quad (4.31)$$

So define

$$\begin{aligned} f_n &\equiv \frac{1}{2}(a_n - ib_n) \\ f_{-n} &\equiv \frac{1}{2}(a_n + ib_n) = f_n^* \\ f_0 &\equiv \frac{a_0}{2}. \end{aligned} \quad (4.32)$$

Then the Fourier series for $F(t)$ can be written as

$$\begin{aligned} F(t) &= f_0 + \sum_{n=1}^{\infty}\left[\, f_n e^{+in\omega t} + f_{-n} e^{-in\omega t}\,\right] \\ &= f_0 + \left(\sum_{n=1}^{\infty} f_n e^{+in\omega t}\right) + \left(\sum_{n=-\infty}^{-1} f_n e^{+in\omega t}\right) \\ F(t) &= \sum_{n=-\infty}^{+\infty} f_n e^{+in\omega t}, \end{aligned} \quad (4.33)$$

were in the second line in the second sum we let $n \rightarrow -n$. Exploiting the formulae for the Fourier coefficients, equation (4.25), we also have for $n = 1, 2, \ldots$

$$\begin{aligned} f_n &= \frac{1}{2}\frac{2}{\tau}\int_{-\frac{\tau}{2}}^{+\frac{\tau}{2}} dtF(t)\left[\cos(n\omega t) - i\sin(n\omega t)\right] = \frac{1}{\tau}\int_{-\frac{\tau}{2}}^{+\frac{\tau}{2}} dtF(t)e^{-in\omega t} \\ f_{-n} &= f_n^* = \frac{1}{2}\frac{2}{\tau}\int_{-\frac{\tau}{2}}^{+\frac{\tau}{2}} dtF(t)\left[\cos(n\omega t) + i\sin(n\omega t)\right] = \frac{1}{\tau}\int_{-\frac{\tau}{2}}^{+\frac{\tau}{2}} dtF(t)e^{+in\omega t} \\ f_0 &= \frac{1}{\tau}\int_{-\frac{\tau}{2}}^{+\frac{\tau}{2}} dtF(t)e^{i0}. \end{aligned} \quad (4.34)$$

Hence we have the general formula for all $n = 0, \pm 1, \pm 2, \ldots$

$$f_n = \frac{1}{\tau}\int_{-\frac{\tau}{2}}^{+\frac{\tau}{2}} dtF(t)e^{-in\omega t}. \quad (4.35)$$

Likewise the particular solution for the coordinate becomes

$$x_p(t) = \frac{a_0}{m}\frac{1}{2\omega_0} + \sum_{n=1}^{\infty}\left\{\frac{1}{2}\left[\hat{a}_n - i\hat{b}_n\right]e^{+in\omega t}e^{-i\delta_n} + \frac{1}{2}\left[\hat{a}_n + i\hat{b}_n\right]e^{-in\omega t}e^{+i\delta_n}\right\}. \tag{4.36}$$

Let, for $n = 1, 2, \ldots$,

$$\begin{aligned} x_0 &\equiv \frac{1}{2}\frac{a_0}{m\omega_0} \\ x_n &\equiv \frac{1}{2}\left(\hat{a}_n - i\hat{b}_n\right)e^{-i\delta_n} \\ x_{-n} &\equiv \frac{1}{2}\left(\hat{a}_n + i\hat{b}_n\right)e^{+i\delta_n} = x_n^*. \end{aligned} \tag{4.37}$$

Hence

$$x_p(t) + x_0 + \sum_{n=1}^{\infty}\left(x_n e^{+in\omega t} + x_{-n}e^{-in\omega t}\right), \tag{4.38}$$

that is

$$x_p(t) = \sum_{n=-\infty}^{+\infty} x_n e^{+in\omega t}. \tag{4.39}$$

Returning to equation (4.28), the coefficients in x_n take on a simpler form as well

$$\begin{aligned} x_0 &= \frac{f_0}{m\omega_0} \\ x_n &= \frac{1}{2}\frac{(a_n - ib_n)}{m}\frac{1}{\sqrt{(\omega_0^2-\omega_n^2)^2 + 4\omega_n^2\beta^2}}e^{-i\delta_n} \\ &= \frac{f_n}{m}e^{-i\delta_n}\frac{1}{\sqrt{(\omega_0^2-\omega_n^2)^2 + 4\omega_n^2\beta^2}} \\ x_{-n} &= x_n^* = \frac{f_{-n}}{m}e^{+i\delta_n}\frac{1}{\sqrt{(\omega_0^2-\omega_n^2)^2 + 4\omega_n^2\beta^2}}, \end{aligned} \tag{4.40}$$

where recall that $f_n^* = f_{-n}$. So to summarize, we have the final form of the Fourier series expansion of the position

$$x_p(t) = \sum_{n=-\infty}^{+\infty} x_n e^{+in\omega t}, \tag{4.41}$$

where

$$x_n = \frac{f_n}{m} e^{-i\delta_n} \frac{1}{\sqrt{(\omega_0^2 - \omega_n^2)^2 + 4\omega_n^2\beta^2}} \tag{4.42}$$

and

$$\tan\delta_n = \left(\frac{2\omega_n\beta}{\omega_0^2 - \omega_n^2}\right) \tag{4.43}$$

with $\delta_{-n} = -\delta_n$.

This can be seen to follow directly from the equations of motion

$$\ddot{x}_p + 2\beta\dot{x}_p + \omega_0^2 x_p = \frac{F(t)}{m}, \tag{4.44}$$

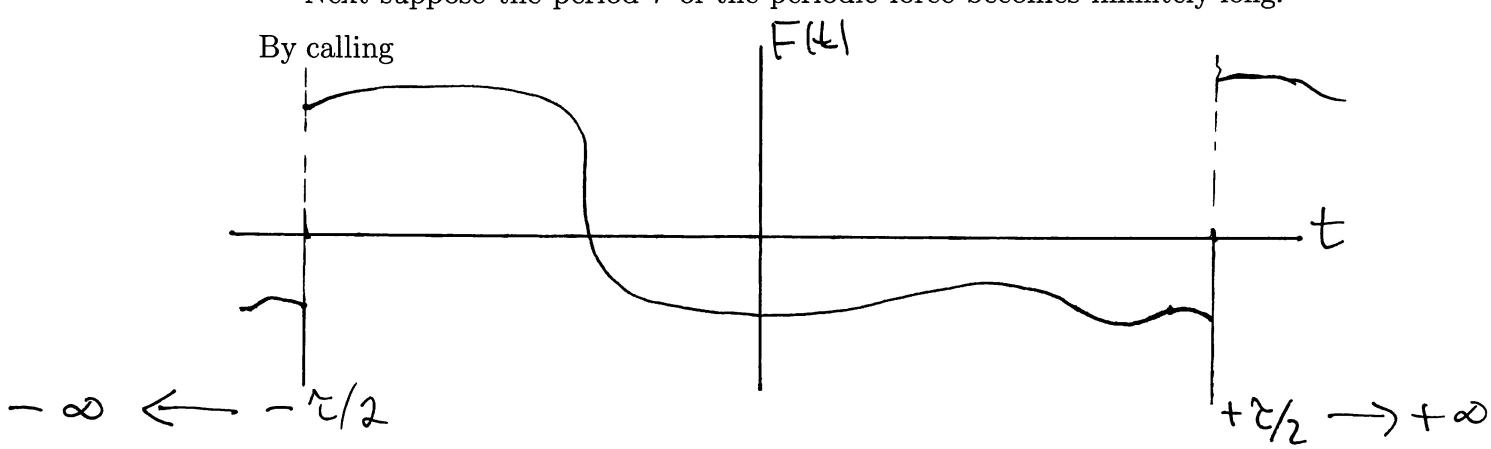
by directly substituting the Fouries series for x_p and F into the equation

$$\sum_{n=-\infty}^{+\infty} e^{+in\omega t}\left\{\left[-\omega_n^2 + 2i\beta\omega_n + \omega_0^2\right] x_n = \frac{f_n}{m}\right\}. \tag{4.45}$$

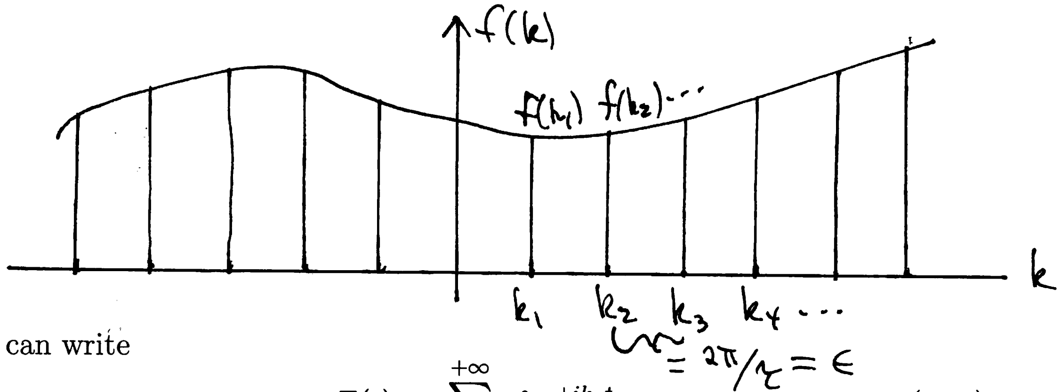
Solving the now algebraic relation yields our previous result

$$\begin{aligned} x_n &= \frac{\frac{f_n}{m}}{[\omega_0^2 - \omega_n^2 + 2i\beta\omega_n]} \\ &= \frac{f_n}{m} e^{-i\delta_n} \frac{1}{\sqrt{(\omega_0^2 - \omega_n^2)^2 + 4\omega_n^2\beta^2}}. \end{aligned} \tag{4.46}$$

Next suppose the period τ of the periodic force becomes infinitely long. By calling



$$n\omega = k_n = \frac{2\pi n}{\tau}, \tag{4.47}$$



we can write

$$F(t) = \sum_{n=-\infty}^{+\infty} f_n e^{+ik_n t} \tag{4.48}$$

and

$$f_n = \frac{1}{\tau} \int_{-\frac{\tau}{2}}^{+\frac{\tau}{2}} dt F(t) e^{-ik_n t}. \tag{4.49}$$

So we can visualize in k-space that

$$\sum_{n=-\infty}^{+\infty} f_n e^{+ik_n t} \tag{4.50}$$

corresponds to an integral of $f(k)e^{ikt}$ where we have divided up the k-axis into intervals of length

$$\epsilon = \frac{2\pi}{\tau}. \tag{4.51}$$

So

$$k_n = n\epsilon \tag{4.52}$$

and

$$\Delta k = k_{n+1} - k_n = \frac{2\pi}{\tau} = \epsilon \tag{4.53}$$

Thus we have

$$\begin{aligned} F(t) &= \sum_{n=-\infty}^{+\infty} \Delta k \frac{\tau}{2\pi} f_n e^{ik_n t} \\ &= \frac{1}{2\pi} \sum_{n=-\infty}^{+\infty} f(k_n) e^{ik_n t} \Delta k, \end{aligned} \tag{4.54}$$

where we have defined

$$f(k_n) = \tau f_n. \tag{4.55}$$

Hence

$$f(k_n) = \tau f_n = \int_{-\frac{\tau}{2}}^{+\frac{\tau}{2}} dt F(t) e^{-ik_n t}. \tag{4.56}$$

So as $\tau \to \infty$ the $\Delta k \to dk$, and we find that

$$\begin{aligned} F(t) &= \int_{-\infty}^{+\infty} \frac{dk}{2\pi} f(k) e^{+ikt} \\ f(k) &= \int_{-\infty}^{+\infty} dt F(t) e^{-ikt}. \end{aligned} \tag{4.57}$$

These are called Fourier Integrals, $f(k) = \tilde{F(k)}$ is called the Fourier transform of $F(t)$ and the $1/2\pi$ factor is convention. So with these notation conventions the Fourier integrals become

$$\begin{aligned} F(t) &= \int_{-\infty}^{+\infty} \frac{dk}{2\pi} \tilde{F}(k) e^{+ikt} \\ \tilde{F}(k) &= \int_{-\infty}^{+\infty} dt F(t) e^{-ikt}. \end{aligned} \tag{4.58}$$

The Fourier transform of the particular solution is given by

$$x_p(t) = \int_{-\infty}^{+\infty} \frac{dk}{2\pi} \tilde{x}_p(k) e^{+ikt}, \tag{4.59}$$

with the Fourier transform function

$$\tilde{x}_p(k) = \frac{\tilde{f}(k)}{m} e^{-i\delta(k)} \frac{1}{\sqrt{(\omega_0^2 - k^2)^2 + 4k^2\beta^2}} \tag{4.60}$$

and

$$\tan\delta(k) = \left(\frac{2k\beta}{\omega_0^2 - k^2} \right). \tag{4.61}$$

Once again this can be obtained directly by substituting the Fourier integral expansions of $x_p(t)$ and $F(t)$ directly into the harmonic oscillator differential equation,

$$\ddot{x}_p + 2\beta\dot{x}_p + \omega_0^2 x_p = \frac{F(t)}{m}, \tag{4.62}$$

to convert it into an algebraic equation

$$\int_{-\infty}^{+\infty} \frac{dk}{2\pi} e^{+ikt} \left\{ \left[-k^2 + 2i\beta k + \omega_0^2\right] \tilde{x}_p(k) = \frac{\tilde{F}(k)}{m} \right\}. \tag{4.63}$$

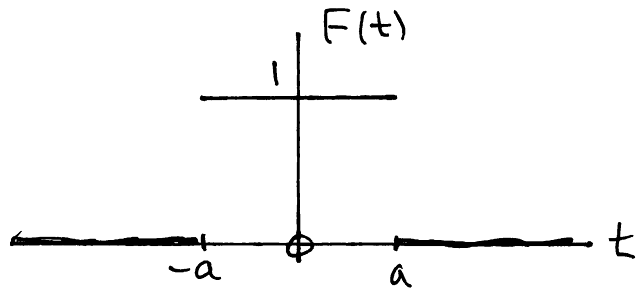
This yields the algebraic solution

$$\begin{aligned}\tilde{x}_p(k) &= \frac{\frac{\tilde{F}(k)}{m}}{[\omega_0^2 - k^2 + 2i\beta k]} \\ &= \frac{\tilde{F}(k)}{m} \underbrace{[\cos\delta(k) - i\sin\delta(k)]}_{=e^{-i\delta(k)}} \frac{1}{\sqrt{(\omega_0^2 - k^2)^2 + 4k^2\beta^2}},\end{aligned} \quad (4.64)$$

with

$$\tan\delta(k) = \left(\frac{2k\beta}{\omega_0^2 - k^2}\right). \quad (4.65)$$

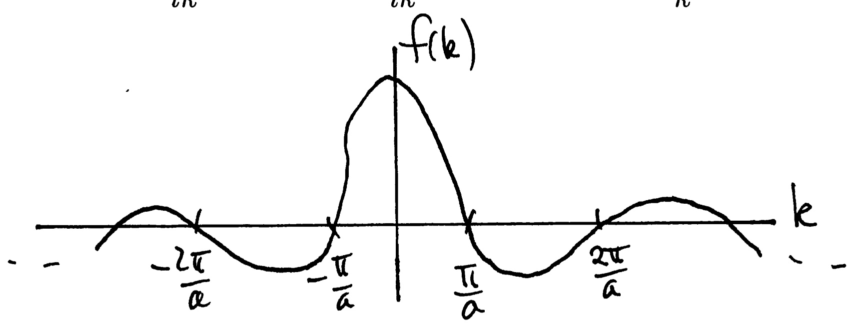
Example: Suppose



$$F(t) = \begin{cases} 1 & |t| < a \\ 0 & |t| > a \end{cases} \quad (4.66)$$

The Fourier transform of the step function is just the diffraction function

$$\begin{aligned}\tilde{F}(k) &= \int_{-\infty}^{+\infty} dt F(t) e^{-ikt} = \int_{-a}^{+a} dt e^{-ikt} \\ &= \frac{1}{-ik} e^{-ikt} \Big|_{t=-a}^{t=+a} = \frac{-1}{ik}\left(e^{-ika} - e^{+ika}\right) = \frac{2}{k}\sin ka.\end{aligned} \quad (4.67)$$



Now notice the property that the Fourier transform of the Fourier transform should equal the original function. This implies that

$$\begin{aligned} F(t) &= \int_{-\infty}^{+\infty} \frac{dk}{2\pi} e^{+ikt} \tilde{F}(k) \\ &= \int_{-\infty}^{+\infty} \frac{dk}{2\pi} e^{+ikt} \left(\int_{-\infty}^{+\infty} dt' e^{-ikt'} F(t') \right) \\ &= \int_{-\infty}^{+\infty} dt' \left(\int_{-\infty}^{+\infty} \frac{dk}{2\pi} e^{-ik(t'-t)} \right) F(t'). \end{aligned} \tag{4.68}$$

In the last line we interchanged the order of integration. Hence since this must again be equal to $F(t)$ we find the Fourier transform of the Dirac delta function

$$\delta(t'-t) = \int_{-\infty}^{+\infty} \frac{dk}{2\pi}\ e^{-ik(t'-t)}. \tag{4.69}$$

Thus the Fourier transform of the Dirac delta function, $\delta(t)$ is just equal to 1, $\tilde{\delta}(k) = 1$

$$\tilde{\delta}(k) = \int_{-\infty}^{+\infty} dt\ \delta(t)\ e^{-ikt} = e^0 = 1. \tag{4.70}$$

A similar Fourier series analysis of the Dirac delta function can be obtained, but due to the required periodicity of the expansion the result is more complicated. Consider the Fourier series for the periodic function $F(t)$

$$\begin{aligned} F(t) &= \sum_{n=-\infty}^{+\infty} f_n e^{+in\omega t} \\ &= \sum_{n=-\infty}^{+\infty} e^{+in\omega t} \left(\frac{1}{\tau} \int_{-\frac{\tau}{2}}^{+\frac{\tau}{2}} dt' e^{-in\omega t'} F(t') \right), \end{aligned} \tag{4.71}$$

interchanging the order of summation and integration, this becomes

$$F(t) = \int_{-\frac{\tau}{2}}^{+\frac{\tau}{2}} dt' \left(\frac{1}{\tau} \sum_{n=-\infty}^{+\infty} e^{-in\omega(t'-t)} \right) F(t'). \tag{4.72}$$

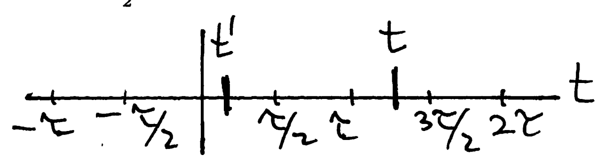
Here we must be a little more careful due to the periodicity of $F(t)$. We note that for $-\tau/2 \le t \le +\tau/2$ we have

$$\frac{1}{\tau}\sum_{n=-\infty}^{+\infty} e^{-in\omega(t'-t)} = \delta(t'-t). \tag{4.73}$$

Next consider $+\tau/2 \le t \le +3\tau/2$, in order for

$$F(t) = \int_{-\frac{\tau}{2}}^{+\frac{\tau}{2}} dt'\Delta(t'-t)F(t') = F(t), \tag{4.74}$$

we must have



$$\Delta(t'-t) = \delta(t'+\tau-t) \tag{4.75}$$

so the argument of δ can vanish as we integrate t' over $(-\tau/2, +\tau/2)$, that is $t = t' + \tau$, and if $-\tau/2 \le t \le +\tau/2$, then $\delta(t'+\tau-t) = 0$. Hence we see that for $-\tau/2 \le t \le +3\tau/2$ we have

$$\frac{1}{\tau}\sum_{n=-\infty}^{+\infty} e^{-in\omega(t'-t)} = \delta(t'-t) + \delta(t'-t+\tau). \tag{4.76}$$

So for t over the entire time range $-\infty < t < +\infty$ we have

$$\frac{1}{\tau}\sum_{n=-\infty}^{+\infty} e^{-in\omega(t'-t)} = \sum_{l=-\infty}^{+\infty} \delta(t'-t+l\tau). \tag{4.77}$$

Now we can check this result with the Fourier transform result as $\tau \to \infty$. Since $(t'-t)$ is finite, only the $l = 0$ term contributes to the sum

$$\begin{aligned} \frac{1}{\tau}\sum_{n=-\infty}^{+\infty} e^{-in\omega(t'-t)} &= \frac{1}{2\pi}\sum_{n=-\infty}^{+\infty} \Delta k e^{-ik_n(t'-t)} \\ &= \int_{-\infty}^{+\infty} \frac{dk}{2\pi} e^{-ik(t'-t)} = \delta(t'-t). \end{aligned} \tag{4.78}$$