2 Conservation Theorems

First consider conservation theorems for single particle motion and then for systems of particles.

1) The first conservation law follows immediately from the 2nd law

$$\vec{F} = \dot{\vec{p}} \tag{2.1}$$

if a particle is **free**, i.e. experiences no net force, then

$$\dot{\vec{p}} = 0 \Rightarrow \vec{p} = \text{const.} \tag{2.2}$$

The linear momentum is said to be **conserved.** Alternatively if $\vec{F} \cdot \vec{s} = 0$ for some constant vector $\vec{s}$, then $\dot{\vec{p}} \cdot \vec{s} = 0$ and $\vec{p} \cdot \vec{s} =$ constant: The linear momentum is conserved in the direction $\vec{s}$ in which the force vanishes.

2) For a single particle we can define its **angular momentum $\vec{L}$ with respect to the origin of our coordinate system** by

$$\vec{L} \equiv \vec{r} \times \vec{p} \tag{2.3}$$

where $\vec{r}$ is the position vector of the particle and $\vec{p}$ is the linear momentum of the particle.

The **torque** $\vec{N}$ with respect to the same origin is defined as

$$\vec{N} \equiv \vec{r} \times \vec{F} = \vec{r} \times \dot{\vec{p}} \tag{2.4}$$

by N 2nd law. Hence

$$\dot{\vec{L}} = \frac{d}{dt}(\vec{r} \times \vec{p}) = \dot{\vec{r}} \times \vec{p} + \vec{r} \times \dot{\vec{p}} \tag{2.5}$$

but

$$\dot{\vec{r}} \times \vec{p} = m\dot{\vec{r}} \times \dot{\vec{r}} = 0 \Rightarrow \dot{\vec{L}} = \vec{r} \times \dot{\vec{p}} = \vec{N} \tag{2.6}$$

Hence, if there are no torques on a particle, $\vec{N} = 0$, then $\dot{\vec{L}} = 0 \Rightarrow \vec{L} = const.$ The angular momentum of a particle subject to no net torque is conserved.

3) Consider the power that is work/time provided by the force $\vec{F}$ acting on the particle

$$\frac{dW}{dt} = \vec{F} \cdot \vec{v} = \dot{\vec{p}} \cdot \vec{v} \tag{2.7}$$

for constant mass

$$\frac{dW}{dt} = m\dot{\vec{v}} \cdot \vec{v} = \frac{1}{2}m\frac{d}{dt}(\vec{v} \cdot \vec{v}) = \frac{d}{dt}(\frac{1}{2}m\vec{v}^{\,2}). \tag{2.8}$$

The work done by the force in the time interval between t and $t + dt$ appears as a change in KE of the particle

$$dW = \vec{F} \cdot \vec{v}dt = d(\frac{1}{2}m\vec{v}^{\,2}) = dT \tag{2.9}$$

with $T \equiv \frac{1}{2}m\vec{v}^{\,2}$. Between t_1 and t_2 we find

$$\int_{t_1}^{t_2} \vec{F} \cdot \vec{v}dt = T_2 - T_1 = \frac{1}{2}m\vec{v}_2^{\,2} - \frac{1}{2}m\vec{v}_1^{\,2}, \tag{2.10}$$

where $\vec{v}_i$ is the velocity of the particle at time t_i.

Further note that $\vec{v}dt = d\vec{r}$, the displacement along the particles trajectory, so if the force is given as a function of position at times along the trajectory $\vec{F}(\vec{r}, t)$ we find that the KE change is given by the line integral of the work

$$T_2 - T_1 = \int_{\vec{r}_1}^{\vec{r}_2} \vec{F} \cdot d\vec{r} \tag{2.11}$$

with $\vec{r}_i$ the position of the particle at time t_i (note: $\vec{F} = \vec{F}(\vec{r}, t(\vec{r}))$ here i.e. we integrate along the trajectory.)

In many cases the line integral of the force around a closed path vanishes

$$\oint_C \vec{F} \cdot d\vec{r} = 0 \tag{2.12}$$

(at a fixed time t, that is $\vec{r}$ and t considered independent here). Then the force is **irrotational** and the line integral

$$\int_{\vec{r}_0}^{\vec{r}} \vec{F}(\vec{r},t) \cdot d\vec{r} \tag{2.13}$$

at **fixed time** is independent of the path of integration, it is only a function of the end points $\vec{r}, \vec{r}_0$. (Since these integrals are evaluated at a given time, they say nothing about what happens in the case of the actual displacement of a particle over the path, unless $\vec{F} = \vec{F}(\vec{r})$ only.) So

$$\int_{\vec{r}_0}^{\vec{r}} \vec{F}(\vec{r},t) \cdot d\vec{r} = -U(\vec{r},t) + U(\vec{r}_0,t) \tag{2.14}$$

with $U(\vec{r}_0, t)$ just a constant of integration. Differentiating with respect to $\vec{R}$, t constant, yields (i.e. let $\vec{r} = \vec{r}_0 + d\vec{r}$, then $\vec{r}_0 \to \vec{r}$)

$$\begin{aligned} \vec{F}(\vec{r},t) \cdot d\vec{r} &= -\frac{\partial U(\vec{r},t)}{\partial x_i} dx_i \\ &= -\vec{\nabla} U(\vec{r},t) \cdot d\vec{r} \end{aligned} \tag{2.15}$$

Since this holds for arbitrary $d\vec{r}$ we have

$$\vec{F}(\vec{r},t) = -\vec{\nabla} U(\vec{r},t) \tag{2.16}$$

as we know since

$$\oint_C \vec{F} \cdot d\vec{r} = 0 \Leftrightarrow \vec{\nabla} \times \vec{F} = 0 \Leftrightarrow \vec{F}(\vec{r},t) = -\vec{\nabla} U(\vec{r},t). \tag{2.17}$$

U is the **potential energy** of the particle is the force field $\vec{F}$. Note that $U + U_0$ (note: U_0 is a constant) yields the same force

$$\vec{F} = -\vec{\nabla} U = -\vec{\nabla}(U + U_0) \tag{2.18}$$

The potential energy has no absolute meaning; only differences in potential energy are physically meaningful i.e. In

$$+U(\vec{r},t) - U(\vec{r}_0,t) = -\int_{\vec{r}_0}^{\vec{r}} \vec{F}\cdot d\vec{r} \tag{2.19}$$

we are free to choose on **reference** point $\vec{r}_0$ for defining potential energies at will. Usually $|\vec{r}_0| \to \infty$ and $U(\vec{r}_0,t) \to 0$, but it depends on the problem under consideration.

Similarly the KE has no absolute meaning since we can choose any inertial frame and $\vec{v}$ is relative to it with $T = \frac{1}{2}m\vec{v}^{\,2}$ which changes from frame to frame. Hence T has no absolute meaning, it is frame dependent.

The sum of KE + PE is called the **total energy** E

$$E = T + U \tag{2.20}$$

So the total time derivative is

$$\begin{aligned}
\frac{dE}{dt} &= \frac{dT}{dt} + \frac{dU}{dt} \\
&= \vec{F}\cdot\dot{\vec{r}} + \frac{\partial U}{\partial t} + \overbrace{\vec{\nabla}U}^{=-\vec{F}}\cdot\frac{d\vec{r}}{dt} \\
&= \vec{F}\cdot\dot{\vec{r}} - \vec{F}\cdot\dot{\vec{r}} + \frac{\partial U}{\partial t} \\
&= \frac{\partial U}{\partial t}
\end{aligned} \tag{2.21}$$

If the force is **conservative** (irrotational and time independent), $\vec{F} = \vec{F}(\vec{r})$, that is **time independent**, then so is U, $\frac{\partial U}{\partial t} = 0$. The **total energy then is conserved** $E = constant$. $\Rightarrow$ **Conservation of Energy:** $T_1 + U_1 = T_2 + U_2$.

To summarize, we have 3 important conservation theorems:

1) Linear momentum is conserved if the particle experiences no net force: $\vec{F} = 0$.

$$\Rightarrow \vec{p} = \text{constant} \tag{2.22}$$

2) Angular momentum is conserved if the particle experiences no net torque: $\vec{N} = 0$.

$$\Rightarrow \vec{L} = \text{constant.} \tag{2.23}$$

3) Total energy is conserved if the Force is conservative:

$$E = \text{constant} \tag{2.24}$$

Although we derived these conservation theorems for a single particle we can proceed similarly for a system of n particles loosely aggregated or forming a rigid body.

The total mass of the system of n particles is denoted by M

$$M = \sum_{\alpha=1}^{n} m_\alpha \tag{2.25}$$

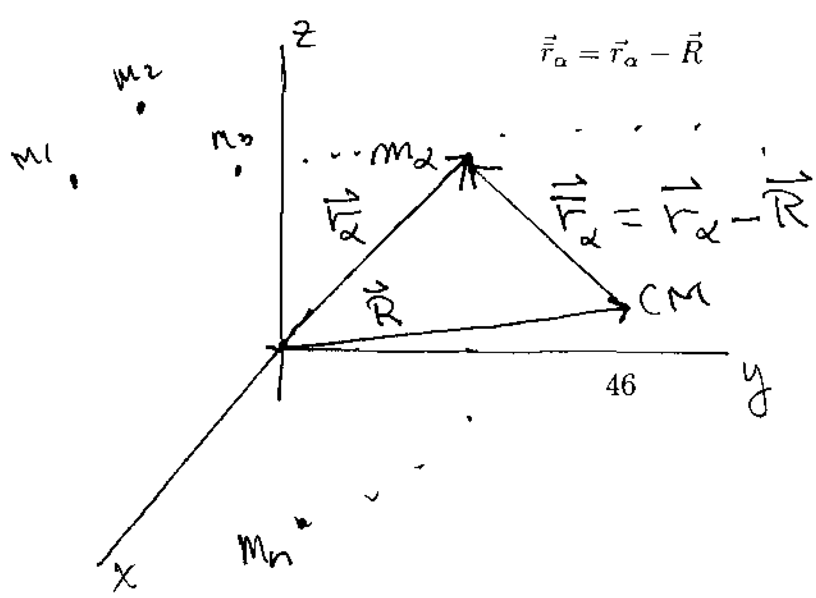
where the subscript α labels the individual particles with masses m_α.

Let the position of the α^{th} particle from the origin of our inertial coordinate system be denoted by $\vec{r}_\alpha$. Then the position, denoted by $\vec{R}$, of the **center-of-mass** of our system is defined to be

$$\vec{R} \equiv \frac{1}{M}\sum_{\alpha=1}^{n} m_\alpha \vec{r}_\alpha \left(= \frac{m_\alpha \vec{r}_\alpha}{M} \right) \tag{2.26}$$

Then the position of the α^{th} particle with respect to the CM is

$$\vec{\bar{r}}_\alpha = \vec{r}_\alpha - \vec{R} \tag{2.27}$$



Taking the n-particles as our system, we have 2 types of forces acting on the particles within the system:

1) The resultant of all forces of external origin to our system:

$$\vec{F}_\alpha^{(e)} = \text{external force on } \alpha^{th} \text{ particle} \tag{2.28}$$

2) The resultant of all forces on particle α arising from the interaction with the $(n-1)$ other particles:

$$\vec{f}_\alpha = \text{ internal force on } \alpha^{th} \text{ particle} \tag{2.29}$$

Further

$$\vec{f}_\alpha = \sum_{\substack{\beta=1 \\ \beta\neq\alpha}}^{n} \vec{f}_{\alpha\beta} \tag{2.30}$$

where $\vec{f}_{\alpha\beta}$ is the internal force on the α^{th} particle due to the β^{th} particle.

The total force acting on the α^{th} particle is

$$\vec{F}_\alpha = \vec{F}_\alpha^{(e)} + \vec{f}_\alpha. \tag{2.31}$$

For internal forces obeying Newton's 3rd law (weak form) we have that

$$\vec{f}_{\alpha\beta} = -\vec{f}_{\beta\alpha} \tag{2.32}$$

(We will also make the stronger assumption that $\vec{f}_{\alpha\beta}$ is a central force (strong form)

$$\vec{f}_{\alpha\beta} = (\vec{r}_\alpha - \vec{r}_\beta)g_{\alpha\beta} \tag{2.33}$$

It will be indicated when this stronger form of N3 is used.)

First $N2 \Rightarrow$ (assuming m_α = constant)

$$\dot{\vec{p}}_\alpha = \frac{d^2}{dt^2}m_\alpha\vec{r}_\alpha = F_\alpha^{(e)} + \sum_{\substack{\beta=1 \\ \beta\neq\alpha}}^{n} \vec{f}_{\alpha\beta} \qquad (\Sigma_\alpha) \tag{2.34}$$

Next we can sum over all α

$$\underbrace{\frac{d^2}{dt^2}\sum_{\alpha=1}^{n} m_\alpha \vec{r}_\alpha}_{=M\vec{R}} = \underbrace{\sum_{\alpha=1}^{n} \vec{F}_\alpha^{(e)}}_{\equiv\vec{F}} + \underbrace{\sum_{\substack{\alpha,\beta=1\\ \alpha\neq\beta}}^{n} \vec{f}_{\alpha\beta}}_{=0} \tag{2.35}$$

where $\vec{F}$ is the sum of all external forces on system, $\vec{F} = \sum_{\alpha=1}^{n} \vec{F}_\alpha^{(e)}$. The last term is zero since

$$\begin{aligned} \sum_{\substack{\alpha,\beta=1\\ \alpha\neq\beta}} \vec{f}_{\alpha\beta} &= \vec{f}_{12} + \vec{f}_{13} + \vec{f}_{23} + \ldots \\ &+ \vec{f}_{21} + \vec{f}_{31} + \vec{f}_{32} + \ldots \\ &= \sum_{\alpha<\beta=1}^{n} (\vec{f}_{\alpha\beta} + \vec{f}_{\beta\alpha}) = 0, \end{aligned} \tag{2.36}$$

where the last line is zero by the weak form of the 3rd law $\vec{f}_{\alpha\beta} = -\vec{f}_{\beta\alpha}$. Hence we have that

$$M\ddot{\vec{R}} = \vec{F} \tag{2.37}$$

The center of mass of a system moves as if it were a single particle, of mass equal to the total mass of the system, acted upon by the total external force, and independent of the nature of the internal forces (as long as they obey N3). The total linear momentum of the system is

$$\vec{P} = \sum_{\alpha=1}^{n} m_\alpha \dot{\vec{r}}_\alpha = \frac{d}{dt}\sum_{\alpha=1}^{n} m_\alpha \vec{r}_\alpha = \frac{d}{dt}(M\vec{R}) \tag{2.38}$$

$$\vec{P} = M\dot{\vec{R}} \tag{2.39}$$

and from above

$$\dot{\vec{P}} = M\ddot{\vec{R}} = \vec{F}. \tag{2.40}$$

1) The total linear momentum of the system is **conserved** if there is no net external force

$$\vec{F} = 0 \Rightarrow \vec{P} = \text{constant} \tag{2.41}$$

2) The total linear momentum of the system is the same as if a single particle of mass M were located at the position of the CM and moving as the CM moves.

The angular momentum of the α^{th} particle with respect to the origin is

$$\vec{L}_\alpha = \vec{r}_\alpha \times \vec{p}_\alpha. \tag{2.42}$$

Hence the total angular momentum of the system about the origin is given by

$$\begin{aligned}
\vec{L} &= \sum_{\alpha=1}^{n} \vec{L}_\alpha = \sum_{\alpha=1}^{n} \vec{r}_\alpha \times \vec{p}_\alpha = \sum_{\alpha=1}^{n} m_\alpha \vec{r}_\alpha \times \dot{\vec{r}}_\alpha \\
&= \sum_{\alpha=1}^{n} m_\alpha (\vec{r}'_\alpha + \vec{R}) \times (\dot{\vec{r}}'_\alpha + \dot{\vec{R}}) \\
&= \sum_{\alpha=1}^{n} m_\alpha [(\vec{r}'_\alpha \times \dot{\vec{r}}') + (\vec{r}'_\alpha \times \dot{\vec{R}}) + (\vec{R} \times \dot{\vec{r}}'_\alpha) + (\vec{R} \times \dot{\vec{R}})]
\end{aligned} \tag{2.43}$$

Note:

$$\sum_{\alpha=1}^{n} m_\alpha \vec{r}'_\alpha \times \dot{\vec{R}} = (\sum_{\alpha=1}^{n} m_\alpha \vec{r}'_\alpha) \times \dot{\vec{R}} \tag{2.44}$$

Now the position of the CM in the CM coordinate system is

$$\sum_{\alpha=1}^{n} m_\alpha \vec{r}'_\alpha = \sum_{\alpha=1}^{n} m_\alpha (\vec{r}_\alpha - \vec{R}) = \sum_{\alpha=1}^{n} m_\alpha \vec{r}_\alpha - M\vec{R} = 0 \tag{2.45}$$

Similarly

$$\sum_{\alpha=1}^{n} m_\alpha (\vec{R} \times \dot{\vec{r}}'_\alpha) = \vec{R} \times \frac{d}{dt} \underbrace{\sum_{\alpha=1}^{n} m_\alpha \vec{r}'_\alpha}_{=0} = 0 \tag{2.46}$$

Thus we secure

$$\vec{L} = \sum_{\alpha=1}^{n} m_\alpha (\vec{\bar{r}}_\alpha \times \dot{\vec{\bar{r}}}_\alpha) + M\vec{R} \times \dot{\vec{R}} \tag{2.47}$$

$$\vec{L} = \sum_{\alpha=1}^{n} \vec{\bar{r}}_\alpha \times \vec{\bar{p}}_\alpha + \vec{R} \times \dot{\vec{P}} \tag{2.48}$$

where

$$\vec{\bar{p}}_\alpha = \vec{p}_\alpha - \left(\frac{m_\alpha}{M}\right)\vec{P} \tag{2.49}$$

i.e.

$$\begin{aligned} \vec{\bar{r}}_\alpha &= \vec{r}_\alpha - \vec{R} \Rightarrow \dot{\vec{\bar{r}}}_\alpha = \dot{\vec{r}}_\alpha - \dot{\vec{R}} \Rightarrow \underbrace{m_\alpha \dot{\vec{\bar{r}}}_\alpha}_{=\vec{\bar{p}}_\alpha} = \underbrace{m_\alpha \dot{\vec{r}}}_{=\vec{p}_\alpha} - m_\alpha \dot{\vec{R}} \\ \Rightarrow&\ \vec{\bar{p}}_\alpha = \vec{p}_\alpha - \left(\frac{m_\alpha}{M}\right)\vec{P}. \end{aligned} \tag{2.50}$$

Thus we find: (the total angular momentum about origin) $\vec{L}$ = (angular momentum of CM about origin) $\vec{R} \times \dot{\vec{P}}$ + (angular momentum of system about CM) $\sum_{\alpha=1}^{n} \vec{\bar{r}}_\alpha \times \vec{\bar{p}}_\alpha$.

Taking the time derivative of the total angular momentum we find

$$\begin{aligned} \dot{\vec{L}} = \sum_{\alpha=1}^{n} \dot{\vec{L}}_\alpha &= \sum_{\alpha=1}^{n} [(\vec{r}_\alpha \times \dot{\vec{p}}_\alpha) + m_\alpha \overbrace{\dot{\vec{r}}_\alpha \times \dot{\vec{r}}_\alpha}^{=0}] \\ &= \sum_{\alpha=1}^{n} \vec{r}_\alpha \times (\vec{F}_\alpha^{(e)} + \sum_{\substack{\beta\neq\alpha \\ \beta=1}}^{n} \vec{f}_{\alpha\beta}) \end{aligned} \tag{2.51}$$

Once again

$$\begin{aligned} \sum_{\substack{\alpha,\beta=1 \\ \alpha\neq\beta}}^{n} \vec{r}_\alpha \times \vec{f}_{\alpha\beta} &= \frac{1}{2}\sum_{\substack{\alpha,\beta=1 \\ \alpha\neq\beta}}^{n} \vec{r}_\alpha \times \vec{f}_{\alpha\beta} + \frac{1}{2}\sum_{\substack{\alpha,\beta=1 \\ \alpha\neq\beta}}^{n} \vec{r}_\beta \overbrace{\times \vec{f}_{\beta\alpha}}^{\text{by weak } N3 = -\vec{f}_{\alpha\beta}} \\ &= \frac{1}{2}\sum_{\substack{\alpha,\beta=1 \\ \alpha\neq\beta}}^{n} (\vec{r}_\alpha - \vec{r}_\beta) \times \vec{f}_{\alpha\beta} \end{aligned} \tag{2.52}$$

Now we impose the **strong form of N3** which states that the interparticle force is also **central**

$$\begin{aligned} \vec{f}_{\alpha\beta} &= (\vec{r}_\alpha - \vec{r}_\beta) g_{\alpha\beta} \\ &\Rightarrow (\vec{r}_\alpha - \vec{r}_\beta) \times \vec{f}_{\alpha\beta} = (\vec{r}_\alpha - \vec{r}_\beta) \times (\vec{r}_\alpha - \vec{r}_\beta) g_{\alpha\beta} = 0 \end{aligned} \tag{2.53}$$

Hence

$$\dot{\vec{L}} = \sum_{\alpha=1}^{n} \vec{r}_\alpha \times \vec{F}_\alpha^{(e)} \tag{2.54}$$

But the **external torque** on the α^{th} particle is $N_\alpha^{(e)}$

$$N_\alpha^{(e)} = \vec{r}_\alpha \times \vec{F}_\alpha^{(e)} \tag{2.55}$$

Hence

$$\dot{\vec{L}} = \sum_{\alpha=1}^{n} N_\alpha^{(e)} = \vec{N}^{(e)} = \text{the total external torque} \tag{2.56}$$

If the net resultant external torques about a given axis vanish, then the total angular momentum of the system about that axis is constant in time (if $\vec{f}_{\alpha\beta}$ obeys strong N3).

Finally we will consider the energy conservation theorem for a system of particles. The total KE of the system is defined as the sum of the individual particle's KE:

$$T \equiv \sum_{\alpha=1}^{n} T_\alpha = \sum_{\alpha=1}^{n} \frac{1}{2} m_\alpha \vec{v}_\alpha^{\,2} \tag{2.57}$$

Since $\vec{r}_\alpha = \vec{\bar{r}}_\alpha + \vec{R}$ each velocity squared becomes

$$\begin{aligned} (\Sigma_\alpha) \qquad \vec{v}_\alpha^{\,2} &= \dot{\vec{r}}_\alpha \cdot \dot{\vec{r}}_\alpha = (\dot{\vec{\bar{r}}}_\alpha + \dot{\vec{R}}) \cdot (\dot{\vec{\bar{r}}}_\alpha + \dot{\vec{R}}) \\ &= (\dot{\vec{\bar{r}}}_\alpha \cdot \dot{\vec{\bar{r}}}_\alpha) + 2(\dot{\vec{\bar{r}}}_\alpha \cdot \dot{\vec{R}}) + \dot{\vec{R}} \cdot \dot{\vec{R}} \\ &= \vec{\bar{v}}_\alpha^2 + 2(\bar{v}_\alpha \cdot \dot{\vec{R}}) + \vec{V}^2, \end{aligned} \tag{2.58}$$

where as before $\vec{v}_\alpha = \dot{\vec{r}}_\alpha$ and $\vec{V} = \dot{\vec{R}}$. So

$$T = \sum_{\alpha=1}^{n} \frac{1}{2} m_\alpha \vec{v}_\alpha^2 + \sum_{\alpha=1}^{n} \frac{1}{2} m_\alpha \vec{V}^2 + [\frac{d}{dt} \underbrace{(\sum_{\alpha=1}^{n} m_\alpha \vec{r}_\alpha)]}_{=0} \cdot \dot{\vec{R}} \tag{2.59}$$

where the last term is zero as it is the location of CM in the CM coordinate system: $\sum_{\alpha=1}^{n} m_\alpha \vec{r}_\alpha = 0$. So finally

$$T = \frac{1}{2} M \vec{V}^2 + \sum_{\alpha=1}^{n} \frac{1}{2} m_\alpha \vec{v}_\alpha{}^2. \tag{2.60}$$

The total KE = (KE of a single particle of mass M moving with the velocity of the CM) + (KE of motion of the individual particles relative to the CM).

The total work done by the total force $\vec{F}_\alpha = \vec{F}_\alpha^{(e)} + \vec{f}_\alpha$ on each particle in going from a configuration 1 of the system to a configuration 2 is just the sum of the individual work integrals:

$$\begin{aligned} W_{1\to2} &= \sum_{\alpha=1}^{n} \int_1^2 \vec{F}_\alpha \cdot \vec{v}_\alpha dt \\ &= \sum_{\alpha=1}^{n} \frac{1}{2} m_\alpha \vec{v}_{2\alpha}^2 - \sum_{\alpha=1}^{n} \frac{1}{2} m_\alpha \vec{v}_{1\alpha}^2 \\ &= T_2 - T_1. \end{aligned} \tag{2.61}$$

As before if the forces are functions of positions $\vec{r}_\alpha$ and time t, we have, using

$$\vec{v}_\alpha dt = d\vec{r}_\alpha, \tag{2.62}$$

that the change in total KE in going from configuration 1 to configuration 2 is

$$T_2 - T_1 = \sum_{\alpha=1}^{n} \int_1^2 \vec{F}_\alpha \cdot d\vec{r}_\alpha$$

$$= \sum_{\alpha=1}^{n} \int_1^2 \vec{F}_\alpha^{(e)} \cdot d\vec{r}_\alpha + \sum_{\alpha=1}^{n} \int_1^2 (\sum_{\substack{\beta=1 \\ \beta \neq \alpha}}^{n} \vec{f}_{\alpha\beta}) \cdot d\vec{r}_\alpha, \qquad (2.63)$$

(where again the line integrals are evaluated along each of the particles trajectories so that if the forces depend upon time, they are evaluatd at the time $t = t(\vec{r}_\alpha)$ along the trajectory and position $\vec{r}_\beta = \vec{r}_\beta(t(\vec{r}_\alpha))$, for example). Further if the **forces** are **irrotational**

$$\vec{F}_\alpha^{(e)} = -\vec{\nabla}_\alpha U_\alpha \qquad (U_\alpha = U_\alpha(\vec{r}_\alpha, t)) \qquad (2.64)$$

$$\vec{f}_{\alpha\beta} = -\vec{\nabla}_\alpha \bar{U}_{\alpha\beta} \quad ; \quad \vec{f}_\alpha = \sum_{\substack{\beta=1 \\ \beta \neq \alpha}}^{n} \vec{f}_{\alpha\beta} \qquad (2.65)$$

and weak N3 is assumed

$$\vec{f}_{\alpha\beta} = -\vec{f}_{\beta\alpha} \qquad (2.66)$$

so the potential depends on the distance between the particles

$$\bar{U}_{\alpha\beta} = \bar{U}_{\alpha\beta}((\vec{r}_\alpha - \vec{r}_\beta), t) \Rightarrow \bar{U}_{\alpha\beta} = \bar{U}_{\beta\alpha} \qquad (2.67)$$

with $\vec{\nabla}_\alpha =$ the gradient operator with respect to $\vec{r}_\alpha$. Hence the total potential energy of the system is just the sum of the potential energy of each particle in the external force field plus the potential energy due to the interparticle forces. Since $\vec{f}_{\alpha\beta}$ is the force on α due to β and by the weak form of $N3$ $\vec{f}_{\beta\alpha} = -\vec{f}_{\alpha\beta}$, the force on β due to α; we only count the **interparticle** potential energy once $U_{\alpha\beta}(\alpha < \beta)$ in the total PE sum. That is the work in assembling our system from a reference point of zero PE (say $\vec{r}_0 = \infty$) to our final particle configuration can be found by bringing in each particle from $\vec{r}_0$ to its final position $\vec{r}_\alpha$; **one at a time**: (recall the work is path independent since the forces are irrotational). (Brought in instantneously with t fixed.)

1) Bring in $\alpha = 1$: The work to do this is just **against** the external force on $\alpha = 1$

$$U^{(1)} = -\int_{\vec{r}_0}^{\vec{r}_1} \vec{F}_1^{(e)} \cdot d\vec{r}_1 \tag{2.68}$$

2) Bring in $\alpha = 2$: The work is just against $\vec{F}_2^{(e)}$ and also $\vec{f}_{21}$ since $\alpha = 1$ is already in place.

$$U^{(2)} = -\int_{\vec{r}_0}^{\vec{r}_2} \vec{F}_2^{(e)} \cdot d\vec{r}_2 + \int_{\vec{r}_0}^{\vec{r}_2} -\vec{f}_{21} \cdot d\vec{r}_2 \tag{2.69}$$

(Note this is the integral along the path particle 2 takes with $\vec{r}_1$ fixed in place, we could use $d\vec{r}_{21} = d\vec{r}_2$.)

3) Bring in $\alpha = 3$

$$U^{(3)} = \int_{\vec{r}_0}^{\vec{r}_3} -\vec{F}_3^{(e)} \cdot d\vec{r}_3 - \int_{\vec{r}_0}^{\vec{r}_3} (+\vec{f}_{31} \cdot d\vec{r}_3 + \vec{f}_{32} \cdot d\vec{r}_3) \tag{2.70}$$

and so on until $\alpha = n$ is brought in to its final position $\vec{r}_n$. Hence the total potential energy U is

$$\begin{aligned} U &= U^{(1)} + U^{(2)} + \ldots + U^{(n)} \\ &= \sum_{\alpha=1}^{n} \int_{\vec{r}_0}^{\vec{r}_\alpha} -\vec{F}_\alpha^{(e)} \cdot d\vec{r}_\alpha + \sum_{\alpha=1}^{n} \sum_{\substack{\beta=1 \\ \alpha>\beta}}^{n} \int_{\vec{r}_0}^{\vec{r}_\alpha} -\vec{f}_{\alpha\beta} \cdot d\vec{r}_\alpha \end{aligned} \tag{2.71}$$

$$U = \sum_{\alpha=1}^{n} U_\alpha + \sum_{\substack{\alpha,\beta-1 \\ \alpha>\beta}} \bar{U}_{\alpha\beta} \tag{2.72}$$

where recall

$$\int_{\vec{r}_0}^{\vec{r}_\alpha} -\vec{F}_\alpha^{(e)} \cdot d\vec{r}_\alpha = \int_{\vec{r}_0}^{\vec{r}_\alpha} \underbrace{\vec{\nabla}_\alpha U_\alpha \cdot d\vec{r}_\alpha}_{=dU_\alpha \text{at fixed } t} = U_\alpha(\vec{r}_\alpha, t) - \not{U}_\alpha(\vec{r}_0, t) \tag{2.73}$$

where our reference potential is taken to vanish $U_\alpha(\vec{r}_0, t) = 0$. Similarly

$$\int_{\vec{r}_0}^{\vec{r}_\alpha} -\vec{f}_{\alpha\beta} \cdot d\vec{r}_\alpha = \int_{\vec{r}_0}^{\vec{r}_{\alpha\beta}} \nabla_\alpha \bar{U}_{\alpha\beta} \cdot d\vec{r}_{\alpha\beta} = \bar{U}_{\alpha\beta} \tag{2.74}$$

Since $\bar{U}_{\alpha\beta} = \bar{U}_{\beta\alpha}$ we may write U as

$$U = \sum_{\alpha=1}^{n} U_\alpha + \sum_{\substack{\alpha,\beta=1 \\ \beta>\alpha}}^{n} \bar{U}_{\alpha\beta} = \sum_{\alpha=1}^{n} U_\alpha + \frac{1}{2} \sum_{\substack{\alpha,\beta=1 \\ \alpha\neq\beta}}^{n} \bar{U}_{\alpha\beta}. \tag{2.75}$$

The total energy of the system is

$$E = T + U, \tag{2.76}$$

hence the total derivative with respect to time is

$$\frac{dE}{dt} = \frac{dT}{dt} + \frac{dU}{dt}. \tag{2.77}$$

As earlier

$$\frac{dT}{dt} = \sum_{\alpha=1}^{n} \vec{F}_\alpha \cdot \vec{v}_\alpha = \sum_{\alpha=1}^{n} \vec{F}_\alpha^{(e)} \cdot \vec{v}_\alpha + \sum_{\alpha=1}^{n} \sum_{\substack{\beta=1 \\ \beta\neq\alpha}}^{n} \vec{f}_{\alpha\beta} \cdot \vec{v}_\alpha \tag{2.78}$$

Using $\vec{f}_{\alpha\beta} = -\vec{f}_{\beta\alpha}$ the last term can be written as

$$\sum_{\alpha=1}^{n} \sum_{\substack{\beta=1 \\ \beta>\alpha}}^{n} \vec{f}_{\alpha\beta} \cdot (\vec{v}_\alpha - \vec{v}_\beta) \tag{2.79}$$

So

$$\frac{dT}{dt} = \sum_{\alpha=1}^{n} \vec{F}_\alpha^{(e)} \cdot \vec{v}_\alpha + \sum_{\alpha=1}^{n} \sum_{\substack{\beta=1 \\ \beta>\alpha}}^{n} \vec{f}_{\alpha\beta} \cdot (\vec{v}_\alpha - \vec{v}_\beta) \tag{2.80}$$

In addition suppose $\vec{F}_\alpha^{(e)} = \vec{F}_\alpha^{(e)}(\vec{r}_\alpha, t)$ so that $U_\alpha = U_\alpha(\vec{r}_\alpha, t)$ while the inter-particle forces $\vec{f}_{\alpha\beta} = \vec{f}_{\alpha\beta}(\vec{r}_\alpha - \vec{r}_\beta, t)$ so that $\bar{U}_{\alpha\beta} = \bar{U}_{\alpha\beta}(\vec{r}_\alpha - \vec{r}_\beta, t)$; then we find

$$\frac{dU}{dt} = \frac{\partial U}{\partial t} + \sum_{\alpha=1}^{n} \overbrace{\vec{\nabla}_\alpha U_\alpha}^{=-\vec{F}_\alpha^{(e)}} \cdot \overbrace{\frac{d\vec{r}_\alpha}{dt}}^{=\vec{v}_\alpha}$$

$$+\sum_{\alpha=1}^{n}\sum_{\substack{\beta=1\\ \beta>\alpha}}^{n}\underbrace{\vec{\nabla}_\beta \bar{U}_{\alpha\beta}}_{=-\vec{f}_{\beta\alpha}}\cdot\frac{\overbrace{d(\vec{r}_\beta-\vec{r}_\alpha)}^{=(\vec{v}_\beta-\vec{v}_\alpha)}}{dt}. \tag{2.81}$$

So

$$\begin{aligned}\frac{dU}{dt} &= \frac{\partial U}{\partial t}-\sum_{\alpha=1}^{n}\vec{F}_\alpha^{(e)}\cdot\vec{v}_\alpha\\ &-\sum_{\alpha=1}^{n}\sum_{\substack{\beta=1\\ \beta>\alpha}}^{n}\underbrace{\vec{f}_{\beta\alpha}}_{=-\vec{f}_{\alpha\beta}}\cdot\overbrace{(\vec{v}_\beta-\vec{v}_\alpha)}^{=-(\vec{v}_\alpha-\vec{v}_\beta)},\end{aligned} \tag{2.82}$$

which yields

$$\begin{aligned}\frac{dU}{dt} &= \frac{\partial U}{\partial t}-\sum_{\alpha=1}^{n}\vec{F}_\alpha^{(e)}\cdot\vec{v}_\alpha\\ &-\sum_{\alpha=1}^{n}\sum_{\substack{\beta=1\\ \beta>\alpha}}^{n}\vec{f}_{\alpha\beta}\cdot(\vec{v}_\alpha-\vec{v}_\beta).\end{aligned} \tag{2.83}$$

Putting all this together

$$\begin{aligned}\frac{dE}{dt} &= \frac{\partial U}{\partial t}-\sum_{\alpha=1}^{n}\vec{F}_\alpha^{(e)}\cdot\vec{v}_\alpha-\sum_{\alpha=1}^{n}\sum_{\substack{\beta=1\\ \beta>\alpha}}^{n}\vec{f}_{\alpha\beta}\cdot(\vec{v}_\alpha-\vec{v}_\beta)\\ &\underbrace{+\sum_{\alpha=1}^{n}\vec{F}_\alpha^{(e)}\cdot\vec{v}_\alpha+\sum_{\alpha=1}^{n}\sum_{\substack{\beta=1\\ \beta>\alpha}}^{n}\vec{f}_{\alpha\beta}\cdot(\vec{v}_\alpha-\vec{v}_\beta)}_{=\frac{dT}{dt}}.\end{aligned} \tag{2.84}$$

Hence

$$\frac{dE}{dt}=\frac{\partial U}{\partial t}. \tag{2.85}$$

If all the forces are conservative $\frac{\partial U}{\partial t}=0$ then $E=$ constant. Then the total energy of the system is conserved $T_1+U_1=T_2+U_2$.

To summarize the 3 conservation laws for a system of particles

1) The total linear momentum of the system is conserved if there is no net total external force acting on the system, $\vec{F} = 0$.

$$\vec{P} = \sum_{\alpha=1}^{n} m_\alpha \dot{\vec{r}}_\alpha = M\dot{\vec{R}} \tag{2.86}$$

and if

$$\vec{F} = \sum_{\alpha=1}^{n} \vec{F}_\alpha^{(e)} = 0 \Rightarrow \vec{P} = \text{constant.} \tag{2.87}$$

2) The total angular momentum of the system is conserved if the net total external torque acting on the system vanishes, $\vec{N} = 0$.

$$\vec{L} = \sum_{\alpha=1}^{n} \vec{r}_\alpha \times \vec{p}_\alpha = \vec{R} \times \vec{P} + \sum_{\alpha=1}^{n} \bar{\vec{r}}_\alpha \times \bar{\vec{p}}_\alpha \tag{2.88}$$

and if

$$\vec{N} = \vec{N}^{(e)} = \sum_{\alpha} \vec{r}_\alpha \times \vec{F}_\alpha^{(e)} = 0 \Rightarrow \vec{L} = \text{constant.} \tag{2.89}$$

3) If all forces (internal and external) are conservative (time independent and irrotational) then the total energy is conserved

$$E = T + U = \text{constant} \tag{2.90}$$

With

$$\begin{aligned} T &= \sum_{\alpha=1}^{n} \frac{1}{2} m_\alpha \vec{v}_\alpha^{\,2} = \frac{1}{2} M \vec{V}^2 + \sum_{\alpha=1}^{n} \frac{1}{2} m_\alpha \bar{\vec{v}}_\alpha^{\,2} \\ U &= \underbrace{\sum_{\alpha=1}^{n} U_\alpha}_{ext.\ PE} + \underbrace{\sum_{\alpha=1}^{n} \sum_{\substack{\beta=1 \\ \beta>\alpha}} \bar{U}_{\alpha\beta}}_{int. PE} \end{aligned} \tag{2.91}$$

Aside: Suppose the force is comprised of conservative plus non-conservative terms (assume single particle mechanics here)

$$\vec{F} = \vec{F}_c + \vec{F}' = -\vec{\nabla} U + \vec{F}' \tag{2.92}$$

Then

$$\begin{aligned}\frac{dW}{dt} &= \vec{F}\cdot\vec{v} = \frac{d}{dt}(\frac{1}{2}m\vec{v}^{\,2}) \\ &= -\vec{\nabla}U\cdot\frac{d\vec{r}}{dt} + \vec{F}'\cdot\vec{v} \qquad (2.93)\\ &= -\frac{(\vec{\nabla}U\cdot d\vec{r})}{dt} + \vec{F}'\cdot\vec{v} \\ &= -\frac{dU}{dt} + \vec{F}'\cdot\vec{v}, \qquad (2.94)\end{aligned}$$

which implies

$$\frac{d}{dt}(\frac{1}{2}m\vec{v}^{\,2} + U) = \frac{d}{dt}(T+U) = \vec{F}'\cdot\vec{v}. \qquad (2.95)$$

Let's go back to the definition of potential energy and discuss more carefully its definition and the work to go from one configuration to another. Now to be more precise the internal forces only depend on the relative coordinate between the two particles so

$$\vec{f}_{\alpha\beta} = \vec{f}_{\alpha\beta}(\vec{r}_\alpha - \vec{r}_\beta, t) \qquad (2.96)$$

from the weak form of N-3 $\vec{f}_{\beta\alpha} = -\vec{f}_{\alpha\beta} \Rightarrow \bar{U}_{\alpha\beta} = \bar{U}_{\beta\alpha}$) So when we integrate along the path in this build up of our system from the, say, infinitely separated reference configurations, to the final configuration we are bringing in the particles one at a time with the previous particles fixed in place so

$$\int_{\vec{r}_{20}}^{\vec{r}_2} -\vec{f}_{21}\cdot d\vec{r}_2 = \int_{\vec{r}_{20}}^{\vec{r}_{21}} -\vec{f}_{21}\cdot d\vec{r}_{21} \qquad (2.97)$$

and hence in general (all at fixed time t)

$$\int_{\vec{r}_{\alpha 0}}^{\vec{r}_\alpha} -\vec{f}_{\alpha\beta}\cdot d\vec{r}_\alpha = \int_{\vec{r}_{\alpha 0}}^{\vec{r}_{\alpha\beta}} -\vec{f}_{\alpha\beta}\cdot d\vec{r}_{\alpha\beta}. \qquad (2.98)$$

So to be more precise we have

$$U = U^{(1)} + U^{(2)} + \ldots + U^{(n)}$$

$$= \sum_{\alpha=1}^{n} \int_{\vec{r}_{\alpha 0}}^{\vec{r}_\alpha} -\vec{F}_\alpha^{(e)} \cdot d\vec{r}_\alpha + \sum_{\alpha=1}^{n} \sum_{\substack{\beta=1 \\ \alpha>\beta}}^{n} \int_{\vec{r}_{\alpha 0}}^{\vec{r}_{\alpha\beta}} -\vec{f}_{\alpha\beta} \cdot d\vec{r}_{\alpha\beta} \qquad (2.99)$$

Now recall

$$\vec{F}_\alpha^{(e)} = \vec{F}_\alpha^{(e)}(\vec{r}_\alpha, t) = -\vec{\nabla}_\alpha U_\alpha \qquad (2.100)$$

with

$$U_\alpha = U_\alpha(\vec{r}_\alpha, t) \qquad (2.101)$$

So

$$dU_\alpha = \vec{\nabla}_\alpha U_\alpha \cdot d\vec{r}_\alpha = -\vec{F}_\alpha^{(e)} \cdot d\vec{r}_\alpha \qquad (2.102)$$

(for fixed t; $\not{\Sigma}_\alpha$). So

$$\int_{\vec{r}_{\alpha 0}}^{\vec{r}_\alpha} -\vec{F}_\alpha^{(e)} \cdot d\vec{r}_\alpha = \int_{U_\alpha(\vec{r}_{\alpha 0}, t)}^{U_\alpha(\vec{r}_\alpha, t)} dU_\alpha = U_\alpha(\vec{r}_\alpha, t) - U_\alpha(\vec{r}_{\alpha 0}, t) \qquad (2.103)$$

So letting $U_\alpha(\vec{r}_{\alpha 0}, t) = 0 \Rightarrow$

$$U = \sum_{\alpha=1}^{n} U_\alpha(\vec{r}_\alpha, t) + \sum_{\alpha=1}^{n} \sum_{\substack{\beta=1 \\ \alpha>\beta}}^{n} \int_{\vec{r}_{\alpha\beta 0}}^{\vec{r}_{\alpha\beta}} -f_{\alpha\beta} \cdot d\vec{r}_{\alpha\beta} \qquad (2.104)$$

(All at time t.)

Now suppose the internal forces are functions of the relative vector $\vec{f}_{\alpha\beta} = \vec{f}_{\alpha\beta}(\vec{r}_\alpha - \vec{r}_\beta, t)$ then

$$\vec{f}_{\alpha\beta} = -\vec{\nabla}_\alpha \bar{U}_{\alpha\beta}(\vec{r}_\alpha - \vec{r}_\beta, t) \qquad (2.105)$$

(Weak $N3 \Rightarrow \bar{U}_{\alpha\beta} = \bar{U}_{\beta\alpha}$) So all at time t ($dt = 0$)

$$\begin{aligned} d\bar{U}_{\alpha\beta} &= \vec{\nabla}_\alpha \bar{U}_{\alpha\beta} \cdot d\vec{r}_\alpha + \vec{\nabla}_\beta \bar{U}_{\alpha\beta} \cdot d\vec{r}_\beta \\ &= \vec{\nabla}_\alpha \bar{U}_{\alpha\beta} \cdot d\vec{r}_\alpha - \vec{\nabla}_\alpha \bar{U}_{\alpha\beta} \cdot d\vec{r}_\beta \\ &= \vec{\nabla}_\alpha \bar{U}_{\alpha\beta} \cdot d\vec{r}_{\alpha\beta} \\ &= -\vec{f}_{\alpha\beta} \cdot d\vec{r}_{\alpha\beta}. \end{aligned} \qquad (2.106)$$

Hence

$$\int_{\vec{r}_{\alpha 0}}^{\vec{r}_{\alpha\beta}} -\vec{f}_{\alpha\beta}\cdot d\vec{r}_{\alpha\beta} = \int_{\bar{U}(\vec{r}_{ref\alpha\beta},t)=0}^{\bar{U}(\vec{r}_{\alpha\beta},t)} = \bar{U}_{\alpha\beta}(\vec{r}_{\alpha\beta},t) - \bar{U}_{\alpha\beta}(\vec{r}_{ref\alpha\beta},t) = \bar{U}_{\alpha\beta}(\vec{r}_{\alpha\beta},t) \tag{2.107}$$

So the total potential energy of the system is

$$U = \sum_{\alpha=1}^{n} U_\alpha(\vec{r}_\alpha,t) + \sum_{\alpha=1}^{n}\sum_{\substack{\beta=1\\ \alpha>\beta}}^{n} \bar{U}_{\alpha\beta}(\vec{r}_{\alpha\beta},t) \tag{2.108}$$

(all at time t). Note the potential energy $\bar{U}_{\alpha\beta}$ is between 2 particles hence

$$\bar{U}_{\alpha\beta} = \bar{U}_{\beta\alpha} \tag{2.109}$$

So we can write

$$U = \sum_{\alpha=1}^{n} U_\alpha + \frac{1}{2}\sum_{\alpha=1}^{n}\sum_{\substack{\beta=1\\ \alpha\neq\beta}}^{n} \bar{U}_{\alpha\beta} \tag{2.110}$$

To see this further, suppose we bring the particles in from the zero of potential energy in the opposite order

$$\begin{aligned} U^{(n)} &= -\int_{\vec{r}_0}^{\vec{r}_n} \vec{F}_n^{(e)}\cdot d\vec{r}_n \\ U^{(n-1)} &= -\int_{\vec{r}_0}^{\vec{r}_{n-1}} \vec{F}_{n-1}^{(e)}\cdot d\vec{r}_{n-1} - \int_{\vec{r}_0}^{\vec{r}_{n-1}} \vec{f}_{n-1n}\cdot d\vec{r}_{n-1} \\ U^{(n-2)} &= -\int_{\vec{r}_0}^{\vec{r}_{n-2}} \vec{F}_{n-2}^{(e)}\cdot d\vec{r}_{n-2} \\ &\quad - \int_{\vec{r}_0}^{\vec{r}_{n-2}} (\vec{f}_{n-2n} + \vec{f}_{n-1n-1})\cdot d\vec{r}_{n-2}, \end{aligned} \tag{2.111}$$

and so on until

$$\begin{aligned} U^{(1)} &= -\int_{\vec{r}_0}^{\vec{r}_1} \vec{F}_1^{(e)}\cdot d\vec{r}_1 \\ &\quad - \int_{r_0}^{\vec{r}_1} (\vec{f}_{1n} + \vec{f}_{1n-1} + \ldots + \vec{f}_{12})\cdot d\vec{r}_1. \end{aligned} \tag{2.112}$$

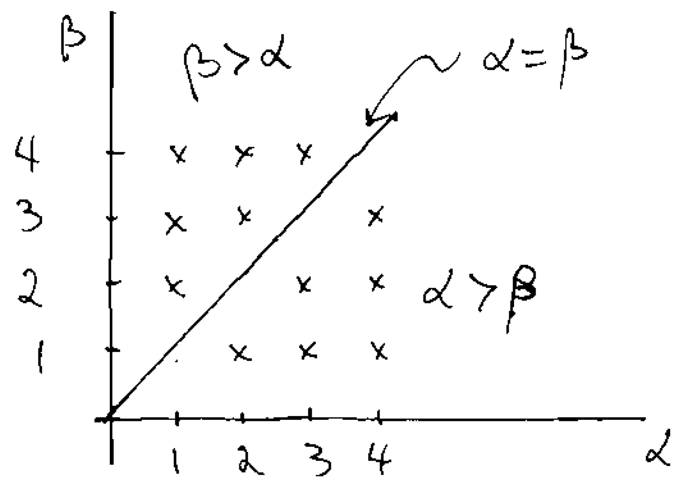
Thus yielding the result

$$\begin{aligned} U &= U^{(n)} + U^{(n-1)} + \dots + U^{(1)} \\ &= \sum_{\alpha=1}^{n} \int_{\vec{r}_{\alpha 0}}^{\vec{r}_\alpha} -\vec{F}_\alpha^{(e)} \cdot d\vec{r}_\alpha + \sum_{\alpha=1}^{n} \sum_{\substack{\beta=1 \\ \beta>\alpha}}^{n} \underbrace{\int_{\vec{r}_{\alpha 0}}^{\vec{r}_\alpha} -\vec{f}_{\alpha\beta} \cdot d\vec{r}_\alpha}_{=\int_{\vec{r}_{\alpha\beta 0}}^{\vec{r}_{\alpha\beta}} -\vec{f}_{\alpha\beta} \cdot d\vec{r}_{\alpha\beta}} . \end{aligned} \quad (2.113)$$

So we have the two expressions for U

$$\begin{aligned} U &= \sum_{\alpha=1}^{n} \int_{\vec{r}_{\alpha 0}}^{\vec{r}_\alpha} -\vec{F}_\alpha^{(e)} \cdot d\vec{r}_\alpha + \sum_{\alpha=1}^{n} \sum_{\substack{\beta=1 \\ \alpha>\beta}}^{n} \int_{\vec{r}_{\alpha 0}}^{\vec{r}_{\alpha\beta}} -\vec{f}_{\alpha\beta} \cdot d\vec{r}_{\alpha\beta} \\ &= \sum_{\alpha=1}^{n} \int_{\vec{r}_{\alpha 0}}^{\vec{r}_\alpha} -\vec{F}_\alpha^{(e)} \cdot d\vec{r}_\alpha + \sum_{\alpha=1}^{n} \sum_{\substack{\beta=1 \\ \beta>\alpha}}^{n} \int_{\vec{r}_{\alpha 0}}^{\vec{r}_{\alpha\beta}} -f_{\alpha\beta} \cdot d\vec{r}_{\alpha\beta} \end{aligned} \quad (2.114)$$

Note $n = 4$



Add 1/2 of each expression (all at time t)

$$U = \sum_{\alpha=1}^{n} \int_{\vec{r}_{\alpha 0}}^{\vec{r}_\alpha} -\vec{F}_\alpha^{(e)} \cdot d\vec{r}_\alpha + \frac{1}{2} \sum_{\alpha=1}^{n} \sum_{\substack{\beta=1 \\ \beta\neq\alpha}}^{n} \int_{\vec{r}_{\alpha 0}}^{\vec{r}_{\alpha\beta}} -\vec{f}_{\alpha\beta} \cdot d\vec{r}_{\alpha\beta}$$

$$= \sum_{\alpha=1}^{n} U_\alpha(\vec{r}_\alpha, t) + \frac{1}{2}\sum_{\alpha=1}^{n}\sum_{\substack{\beta=1\\ \beta\neq\alpha}}^{n} \bar{U}_{\alpha\beta}(\vec{r}_{\alpha\beta}, t) \tag{2.115}$$

Now when going from configuration 1 to configuration 2 this takes *time* $(t_1 - t_2)$. The work done is the difference in KE: $T_2 - T_1$. If the forces are *conservative*, then we can relate this to the difference in potential energies

$$T_2 - T_1 = \sum_{\alpha=1}^{n} \int_{\vec{r}_{1\alpha}}^{\vec{r}_{2\alpha}} \vec{F}_\alpha^{(e)} \cdot d\vec{r}_\alpha + \sum_{\alpha=1}^{n}\sum_{\substack{\beta=1\\ \beta\neq\alpha}}^{n} \int_1^2 \vec{f}_{\alpha\beta} \cdot d\vec{r}_\alpha, \tag{2.116}$$

where configuration 1 is at time $t = t_1$ and configuration 2 is at time $t = t_2$, all the particles move along their respective paths as the time evolves from t_1 to t_2. Now

$$\begin{aligned} \sum_{\alpha=1}^{n}\sum_{\beta=1}^{n} \int_1^2 \vec{f}_{\alpha\beta} \cdot d\vec{r}_\alpha &= \sum_{\alpha=1}^{n}\sum_{\substack{\beta=1\\ \alpha<\beta}}^{n} \int_1^2 [\vec{f}_{\alpha\beta} \cdot d\vec{r}_\alpha + \vec{f}_{\beta\alpha} \cdot d\vec{r}_\beta] \\ &= \sum_{\alpha=1}^{n}\sum_{\substack{\beta=1\\ \alpha<\beta}}^{n} \int_1^2 \vec{f}_{\alpha\beta} \cdot d\vec{r}_{\alpha\beta}. \end{aligned} \tag{2.117}$$

If $\vec{f}_{\alpha\beta}$ is independent of time

$$= -\sum_{\alpha=1}^{n}\sum_{\substack{\beta=1\\ \alpha<\beta}}^{n} [\bar{U}_{\alpha\beta}(\vec{r}_{2\alpha\beta}) - \bar{U}_{\alpha\beta}(\vec{r}_{1\alpha\beta})] \tag{2.118}$$

Likewise for

$$-\int_{\vec{r}_{1\alpha}}^{\vec{r}_{2\alpha}} \vec{F}_\alpha^{(e)} \cdot d\vec{r}_\alpha = U_\alpha(\vec{r}_{2\alpha}) - U_\alpha(\vec{r}_{1\alpha}) \tag{2.119}$$

if $\vec{F}_\alpha^{(e)}$ is independent of time. So

$$T_2 - T_1 = -U_2 + U_1 \tag{2.120}$$

or

$$E_1 = T_1 + U_1 = T_2 + U_2 = E_2 \equiv E \tag{2.121}$$

the total energy. So the total energy is conserved if all forces are conservative.

Now suppose $\vec{F}_\alpha^{(e)}$ and $\vec{f}_{\alpha\beta}$ depend on time then

$$\begin{aligned}
dU_\alpha(\vec{r}_\alpha, t) &= \vec{\nabla}_\alpha U_\alpha \cdot d\vec{r}_\alpha + \frac{\partial U_\alpha}{\partial t}dt = -\vec{F}_\alpha^{(e)} \cdot d\vec{r}_\alpha + \frac{\partial U_\alpha}{\partial t}dt \\
d\bar{U}_{\alpha\beta}(\vec{r}_{\alpha\beta}, t) &= \vec{\nabla}_\alpha \bar{U}_{\alpha\beta} \cdot d\vec{r}_\alpha + \vec{\nabla}_\beta \bar{U}_{\alpha\beta} \cdot d\vec{r}_\beta + \frac{\partial \bar{U}_{\alpha\beta}}{\partial t}dt \\
(\text{weak } N3 \Rightarrow) &= \vec{\nabla}_\alpha \bar{U}_{\alpha\beta} \cdot d\vec{r}_{\alpha\beta} + \frac{\partial \bar{U}_{\alpha\beta}}{\partial t}dt \\
&= -\vec{f}_{\alpha\beta} \cdot d\vec{r}_{\alpha\beta} + \frac{\partial \bar{U}_{\alpha\beta}}{\partial t}dt. \qquad (2.122)
\end{aligned}$$

This implies

$$T_2 - T_1 = -(U_2 - U_1) + \sum_{\alpha=1}^{n} \int_{t_1}^{t_2} \frac{\partial U_\alpha}{\partial t}dt + \sum_{\alpha=1}^{n} \sum_{\substack{\beta=1 \\ \beta>\alpha}}^{n} \int_{t_1}^{t_2} \frac{\partial \bar{U}_{\alpha\beta}}{\partial t}dt \qquad (2.123)$$

So with the total energy defined as $E \equiv T + U$, this yields

$$\begin{aligned}
E_2 - E_1 &= \sum_{\alpha=1}^{n} \int_{t_1}^{t_2} \frac{\partial U_\alpha}{\partial t}dt + \sum_{\alpha=1}^{n} \sum_{\substack{\beta=1 \\ \alpha<\beta}}^{n} \int_{t_1}^{t_2} \frac{\partial \bar{U}_{\alpha\beta}}{\partial t}dt \\
E_2 - E_1 &= \int_{t_1}^{t_2} \frac{\partial U}{\partial t}dt \qquad (2.124)
\end{aligned}$$

or differentially

$$\frac{dE}{dt} = \frac{\partial U}{\partial t}. \qquad (2.125)$$