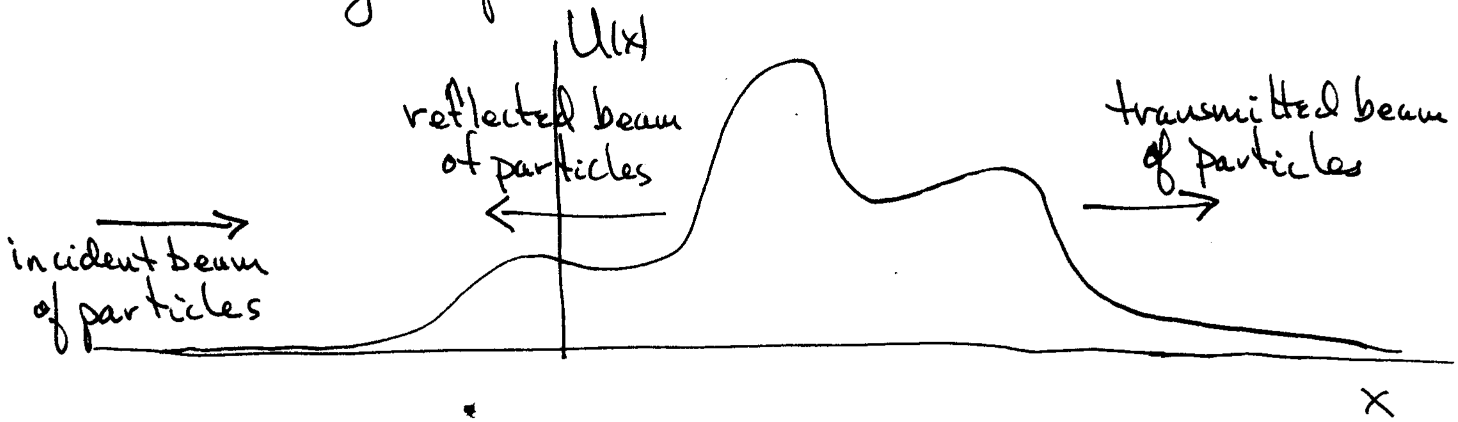
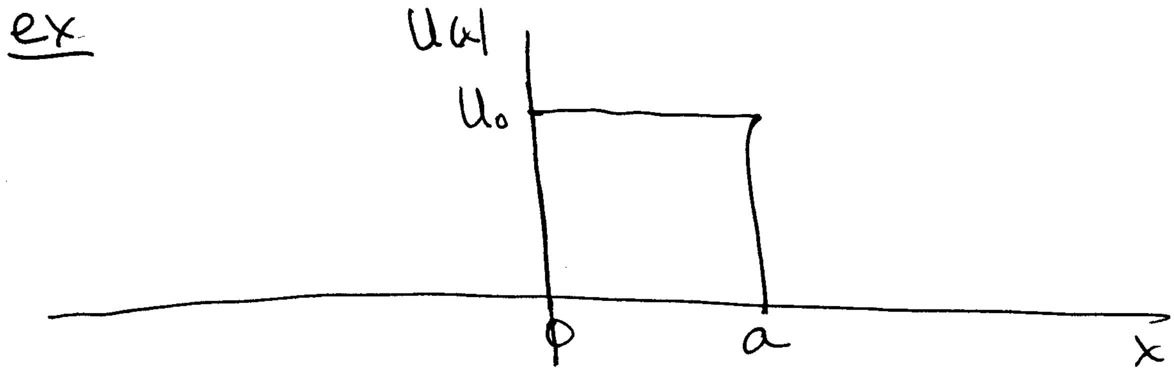
Scattering experiment:



ex



$$U(x) = \begin{cases} 0 & \text{for } x<0 \\ U_0 & \text{for } 0 \le x \le a \\ 0 & \text{for } x>0 \end{cases}$$

Incident wavefunction: $\Psi_{inc}(x,t) = \psi_{inc}(x) e^{-i\frac{E}{\hbar}t}$

$$\psi_{inc}(x) = N e^{ikx}$$

plane wave propagating in the + x direction

i.e. $\hat{P}\Psi_{inc} = +\hbar k \Psi_{inc}$

$\Rightarrow E = \frac{\hbar^2 k^2}{2m}$ the energy of the incident beam of particles

Besides incident wavefunction there is superimposed the scattered wavefunction so that $\Psi = \Psi_{inc} + \Psi_{scattered}$.

$\Psi_{scattered}$ has a piece that is reflected back to the left, $\Psi_{refl.}$, and a piece that is transmitted to the right, Ψ_{trans}.

In the lab we measure the ratio of the scattered probability fluxes to the incident probability flux.

Probability current density

$$S = \frac{\hbar}{2im}\left(\Psi^* \frac{d\Psi}{dx} - \frac{d\Psi^*}{dx}\Psi \right)$$

gives the prob.

$$S_{inc} = \frac{\hbar}{2im}\left[\Psi^*_{inc} \frac{d\Psi_{inc}}{dx} - \frac{d\Psi^*_{inc}}{dx}\Psi_{inc} \right]$$

$$= |N|^2 \frac{\hbar k}{m} = \underbrace{\frac{\hbar k}{m}}_{\substack{\text{velocity}\\ \text{of free}\\ \text{particle}}} \underbrace{\rho_{inc}}_{\substack{\text{prob.}\\ \text{density}}} \; ; \; \rho_{inc} = |\Psi_{inc}|^2 = |N|^2$$

So we measure the reflection coefficient

$$R \equiv \left| \frac{S_{\text{reflected}}}{S_{\text{incident}}} \right|$$

and the transmission coefficient

$$T \equiv \left| \frac{S_{\text{transmitted}}}{S_{\text{incident}}} \right|$$

where

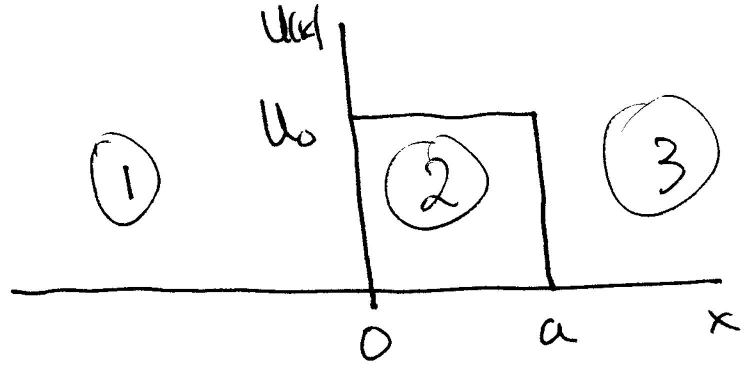
$$S_{\text{refl.}} = \frac{\hbar}{2im}\left[\psi^*_{\text{refl.}} \frac{d\psi_{\text{refl.}}}{dx} - \frac{d\psi^*_{\text{refl.}}}{dx}\psi_{\text{refl.}} \right]$$

$$S_{\text{trans}} = \frac{\hbar}{2im}\left[\psi^*_{\text{trans}} \frac{d\psi_{\text{trans}}}{dx} - \frac{d\psi^*_{\text{trans}}}{dx}\psi_{\text{trans}} \right]$$

So we must find $\psi_{\text{ref.}}$ & ψ_{trans} to predict R, T.

<u>Example</u>: Scattering of a particle of mass m and energy $E>0$ off a potential barrier $U(x)$ above:

3 regions



$U=0$
$x<0$
$x>a$

$$-\frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2}=E\psi$$

$U=U_0$
$0\le x\le a$

$$-\frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2}+U_0\psi=E\psi$$

<u>2 Cases</u>: $E>U_0$, $E<U_0$

1) $E>U_0$: Define

$$k\equiv\sqrt{\frac{2Em}{\hbar^2}}>0\quad\left(E=\frac{\hbar^2k^2}{2m}\right)$$

$$q\equiv\sqrt{\frac{2m(E-U_0)}{\hbar^2}}>0$$

$\Rightarrow$ $x<0$, $x>a$

$$\frac{d^2}{dx^2}\psi+k^2\psi=0$$

$0\le x\le a$

$$\frac{d^2}{dx^2}\psi+q^2\psi=0$$

Solutions:

$$\psi(x) = N \begin{cases} e^{ikx} + r e^{-ikx} & \text{for } x<0 \\ A e^{iqx} + B e^{-iqx} & \text{for } 0 \le x \le a \\ t e^{ikx} + C e^{-ikx} & \text{for } x>a \end{cases}$$

Now e^{ikx} corresponds to a wave travelling in the $+x$ direction

e^{-ikx} corresponds to a wave travelling in the $-x$ direction

Interpretation:

Hence the incident particle's wavefunction is

$$\psi_{inc} = N e^{ikx}$$

, it is coming in from $x \to -\infty$ direction.

The reflected wavefunction corresponds to $\psi - \psi_{inc}$ on the left side of the potential

$$\psi_{refl.} = (\psi - \psi_{inc})\Big|_{x<0} = N r e^{-ikx},$$

travelling back to $x \to -\infty$.

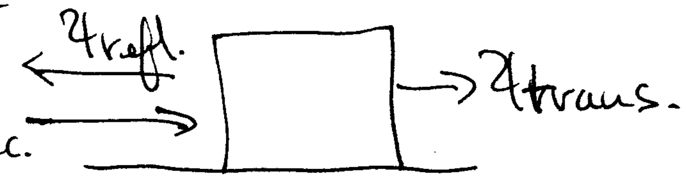
The transmitted wave corresponds to a right travelling wave on the right side of the potential

$$\psi_{trans} = N t e^{+ikx} .$$

Since we prepared the beam of particles to be coming in from the left there should be no wave travelling back to the left on the right of the potential, hence we demand the B.C.

$$C = 0 ,$$ corresponding

to our experiment



Hence we must determine A, B, r, t from applying boundary conditions as we go from one region to the other; at $x = 0$ and $x = a$.

Recall ψ is continuous
$\frac{d\psi}{dx}$ is continuous (since U is finite)

1) $x=0$

$$\boxed{\begin{aligned} \psi(x=0^-) &= (1+r)N \\ &\;\;\shortparallel \qquad\quad \shortparallel \\ \psi(x=0^+) &= (A+B)N \end{aligned}}$$

$$\boxed{\begin{aligned} \frac{d\psi}{dx}(x=0^-) &= k(1-r)Ni \\ &\;\;\shortparallel \qquad\quad \shortparallel \\ \frac{d\psi}{dx}(x=0^+) &= q(A-B)Ni \end{aligned}}$$

$$\Rightarrow \quad \boxed{\begin{aligned} A &= \tfrac{1}{2}\left(1+\tfrac{k}{q}\right) + \tfrac{r}{2}\left(1-\tfrac{k}{q}\right) \\ B &= \tfrac{1}{2}\left(1-\tfrac{k}{q}\right) + \tfrac{r}{2}\left(1+\tfrac{k}{q}\right) \end{aligned}}$$

2) $x=a$

$$\boxed{\begin{aligned} \psi(x=a^-) &= (Ae^{iqa}+Be^{-iqa})N \\ &\;\;\shortparallel \qquad\quad \shortparallel \\ \psi(x=a^+) &= te^{ika}N \end{aligned}}$$

$$\frac{d\psi}{dx}(x=a^-) = N\, qi\left[Ae^{iqa} - Be^{-iqa}\right]$$

$$\| \qquad\qquad\qquad \|$$

$$\frac{d\psi}{dx}(x=a^+) = N t i k e^{ika}$$

$\Rightarrow$

$$\boxed{\begin{aligned} A &= \tfrac{1}{2}\left(1+\tfrac{k}{q}\right)t e^{i(k-q)a} \\ B &= \tfrac{1}{2}\left(1-\tfrac{k}{q}\right)t e^{i(k+q)a} \end{aligned}}$$

Eliminate A&B from above $\Rightarrow$

$$\boxed{\begin{aligned} r &= \frac{i(q^2-k^2)\sin qa}{2kq\cos qa - i(q^2+k^2)\sin qa} \\ t &= \frac{2kq\, e^{-ika}}{2kq\cos qa - i(q^2+k^2)\sin qa} \end{aligned}}$$

Now

$$S_{refl} = -\frac{\hbar k}{m}|r|^2|N|^2 \qquad S_{inc} = \frac{\hbar k}{m}|N|^2$$

$$S_{trans} = \frac{\hbar k}{m}|t|^2|N|^2$$

Hence

$$R = \left|\frac{S_{refl.}}{S_{inc}}\right| = |r|^2 = \frac{(q^2-k^2)^2 \sin^2 qa}{4k^2q^2 + (q^2-k^2)^2 \sin^2 qa}$$

$$T = \left|\frac{S_{trans}}{S_{inc}}\right| = |t|^2 = \frac{4k^2q^2}{4k^2q^2 + (q^2-k^2)^2 \sin^2 qa}$$

Remarks:

1) R, T indep. of normalization of incident wave N.

2) $$R + T = 1 \quad \text{conservation of prob.}$$

3) $$k^2 = \frac{2mE}{\hbar^2}, \quad q^2 = \frac{2m(E-U_0)}{\hbar^2}$$

$\Rightarrow$

$$R = \frac{U_0^2 \sin^2\left(\sqrt{\frac{2m(E-U_0)}{\hbar^2}}\, a\right)}{4E(E-U_0) + U_0^2 \sin^2\left(\sqrt{\frac{2m(E-U_0)}{\hbar^2}}\, a\right)}$$

$$T = \frac{4E(E-U_0)}{4E(E-U_0) + U_0^2 \sin^2\left(\sqrt{\frac{2m(E-U_0)}{\hbar^2}}\, a\right)}$$

Plot T: 1) maxima at $\sqrt{\frac{2m(E-U_0)}{\hbar^2}}\, a = n\pi$

$n = 0, 1, 2, \ldots$ where $T = 1$ $(R = 0)$

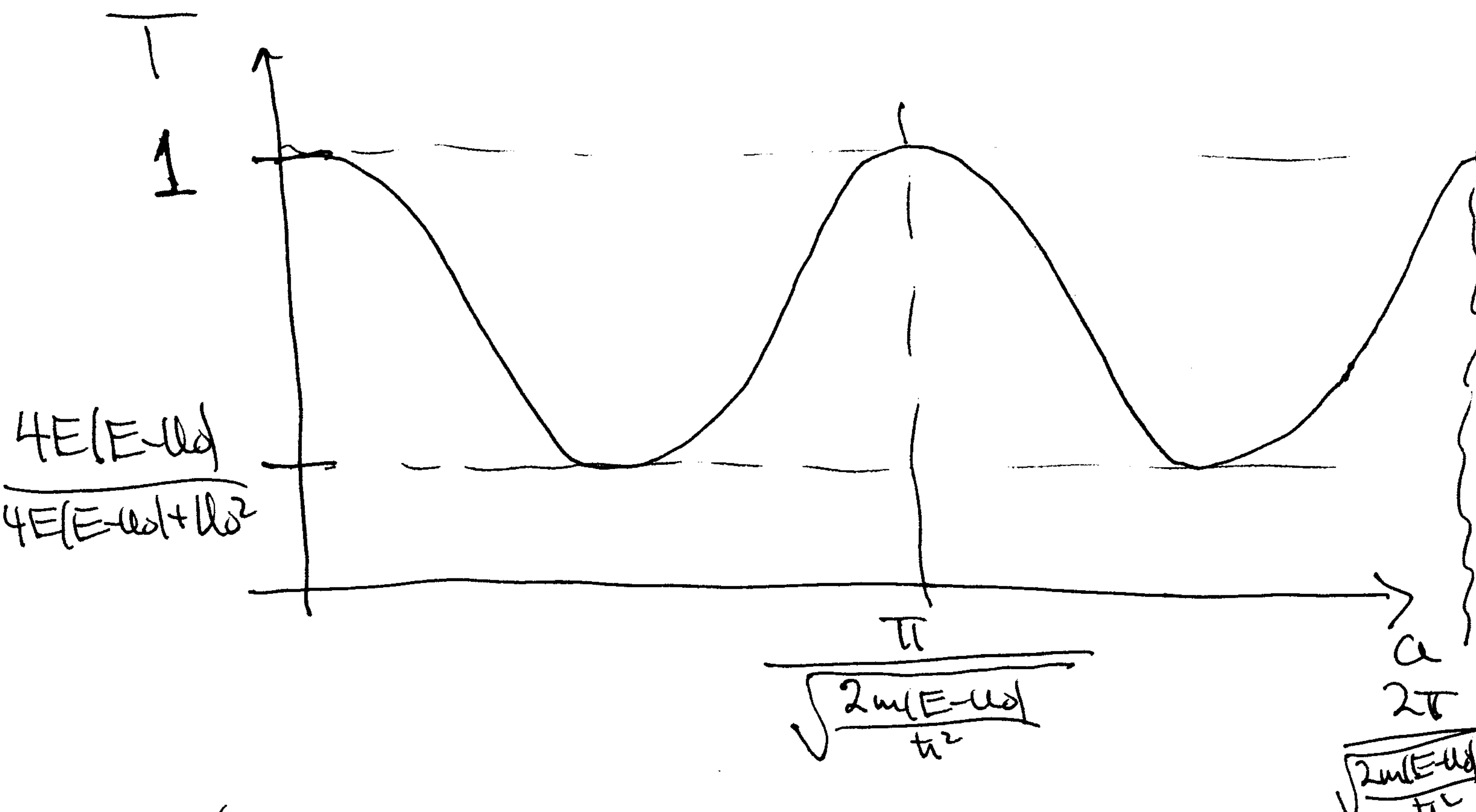
2) minima at $\sqrt{\frac{2m(E-U_0)}{\hbar^2}}\, a = (n+\frac{1}{2})\pi$, $n = 0, 1, 2, \ldots$

where

$$T = \frac{4E(E-U_0)}{4E(E-U_0) + U_0^2}$$

&

$$R = 1 - T = \frac{U_0^2}{4E(E-U_0) + U_0^2}$$



(T=1 particles said to undergo "resonance scattering")

Case 2) $V_0 > E > 0$ Barrier Penetration (Tunneling)

Define: • $k \equiv \sqrt{\frac{2mE}{\hbar^2}} > 0$ (same)

but now

$$\kappa \equiv \sqrt{\frac{2m(U_0 - E)}{\hbar^2}} > 0$$

The time indep Schrödinger equation

$x < 0$
$x > a$ $\quad \frac{d^2}{dx^2}\psi + k^2\psi = 0$

$0 \le x \le a$ $\quad \frac{d^2}{d^2x}\psi - \kappa^2\psi = 0$

minus

Proceeding as before except for $0 \le x \le a$:

$$\psi(x) = N \begin{cases} e^{ikx} + re^{-ikx} & \text{for } x<0 \\ Ae^{+\kappa x} + Be^{-\kappa x} & \text{for } 0 \le x \le a \\ te^{ikx} & \text{for } x>a \end{cases}$$

This is just as if we let $q = -i\varkappa$ in the first case! Since ψ and $\frac{d\psi}{dx}$ are still continuous we can obtain Case 2 by the substitution (trick) $q \to -i\varkappa$

Recall $\quad \sin qa = \sin(-i\varkappa a) = -i \sinh \varkappa a$

hence $\quad q^2 = -\varkappa^2$

$$\sin^2 qa = -\sinh^2 \varkappa a$$

and we have the transmission & reflection coefficients for case 2

$$\boxed{\begin{aligned} T &= \frac{4k^2\varkappa^2}{4k^2\varkappa^2 + (\varkappa^2+k^2)^2 \sinh^2(\varkappa a)} \\ &= \frac{4E(U_0-E)}{4E(U_0-E) + U_0^2 \sinh^2\left(\sqrt{\frac{2m(U_0-E)}{\hbar^2}}\, a\right)} \end{aligned}}$$

$(R = 1 - T)$

Note: for $\kappa a = \sqrt{\frac{2m(U_0-E)}{\hbar^2}}\, a >> 1$

Then $\sinh^2(\kappa a) = \frac{1}{4}(e^{\kappa a} - e^{-\kappa a})^2 \approx \frac{e^{2\kappa a}}{4}$

and

$$\boxed{T \approx \frac{16E(U_0-E)}{U_0^2}\, e^{-2\sqrt{\frac{2m(U_0-E)}{\hbar^2}}\, a}}$$

$$\boxed{\neq 0}$$!! (Classically T=0)

The particle is said to have penetrated the barrier or tunneled through the barrier. Ex. Tunnel diode, α-decay of nuclei, scanning tunneling microscope, Josephson effect etc.

Ex.1) e^- : $\frac{1}{\kappa} = \frac{\hbar}{\sqrt{2m(U_0-E)}} = \frac{0.196\text{ nm}}{\sqrt{U_0-E}}$

with U_0 & E in eV.

Typical atomic physics values $E = 1\text{eV}$

$U_0 = 2\text{eV}$

$a = 0.1\text{ nm}$

$\Rightarrow \kappa a = \frac{1}{1.96} \Rightarrow \boxed{T = 0.78}$

Quite a large prob. for e^- to tunnel through the barrier.

Ex 2) protons incident instead of e^-

$T << 1$ since $\frac{m_{e^-}}{m_p} \triangleq \frac{1}{1836}$

$$\Rightarrow \quad \frac{1}{\chi} = \frac{0.196\ \text{nm}}{\sqrt{1836}\sqrt{U_0 - E}} = \frac{4.6 \times 10^{-3}\ \text{nm}}{\sqrt{U_0 - E}}$$

with E, U_0 in eV. For the above values for E, U_0, a

$$\Rightarrow \kappa a = \frac{\sqrt{1836}}{1.96} \approx 22 >> 1$$

$$\Rightarrow \quad T \approx 4 \times 10^{-19}$$, quite negligible!