- VI.G) 3) $\Rightarrow$ $$\boxed{B = \frac{\mu\mu_0 N i}{\mu l + \mu_0(2\pi r - l)}}$$

for ferromagnetic materials $(\mu = \mu(H))$ $\mu l >> 2\pi r$

$$B \approx \frac{\mu_0 N i}{l} >> \frac{\mu_0 N i}{2\pi r}$$

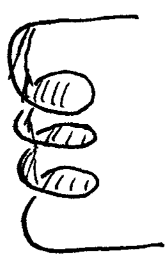
VII) Inductance:

A) Self inductance: Faraday's law $\mathcal{E}_L = -\frac{d\Phi_B}{dt} = -L\frac{di}{dt}$

$$L = \frac{d\Phi_B}{di} = \frac{\Phi_B}{i} \text{ for linear media.}$$

1) SI unit of inductance = henry $(H) \equiv 1 \frac{\text{Volt}\cdot\text{sec.}}{\text{Ampere}}$

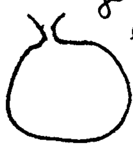
2) Solenoid: a) flux linkages = total flux thru solenoid.



$$\Phi_B = nl\,\Phi_{B\,coil} = nl\,B\pi r^2 = \mu_0 n^2 l A i$$

$$\Rightarrow L = \mu_0 n^2 l A$$

b) Voltage across each coil $= \mathcal{E}_{coil}$

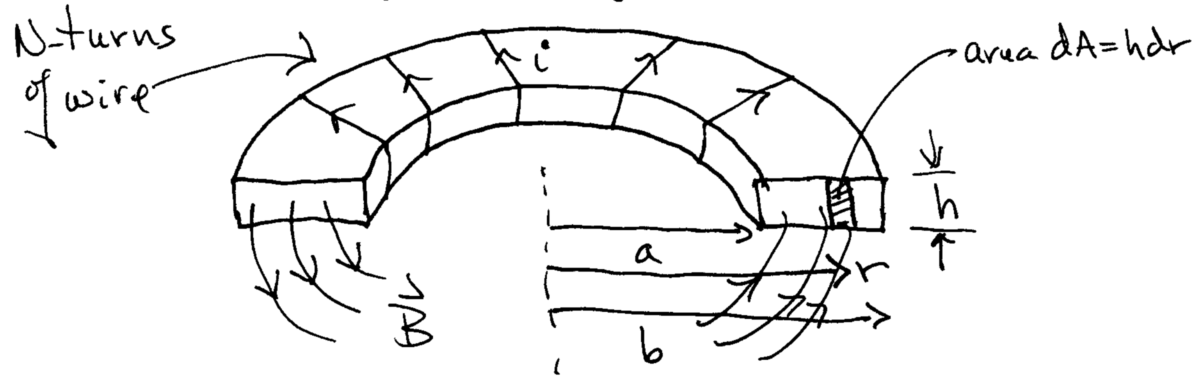


$$\mathcal{E}_{coil} = -\frac{d\Phi_{B\,coil}}{dt} = -\mu_0 n \pi r^2 \frac{di}{dt}$$

But there are nl-coils $\Rightarrow \mathcal{E}_L = nl\cdot\mathcal{E}_{coil}$

$$\Rightarrow L = \mu_0 n^2 l A$$

VII. A) 3) Toroid of rectangular cross section:



Total flux through coil of wire $\phi_B = N \int_{1\,coil} \vec{B} \cdot d\vec{A}$

$$\phi_B = N \int_{r=a}^{b} B(r)\, h\, dr$$

Use Ampere's law to find $B(r)$

$$\underbrace{\oint_C \vec{B} \cdot d\vec{s}}_{B(r)\,2\pi r} = \mu_0\, i_{enclosed} = \mu_0 N i \quad \Rightarrow B(r) = \frac{\mu_0 N i}{2\pi r}$$

So $\phi_B = \dfrac{\mu_0 N^2 i h}{2\pi} \ln\left(\dfrac{b}{a}\right) i$

$$\Rightarrow \boxed{L = \frac{\mu_0 N^2 h}{2\pi} \ln\left(\frac{b}{a}\right)}$$

---

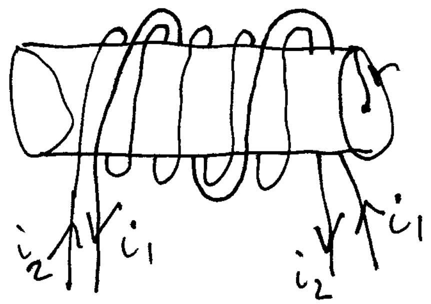
4) Fill inductors with magnetic material

$$\vec{B} = \frac{\mu}{\mu_0} \vec{B}_0 = (1 + \chi_m) \vec{B}_0$$ ← $\vec{B}$ without magnetic material

- VII.A.4) $\Rightarrow$ Solenoid: $L = \mu n^2 A l$

Toroid: $L = \frac{\mu N^2 h}{2\pi} \ln\left(\frac{b}{a}\right)$

B) Mutual Inductance: Wrap 2 independent coils of wire around solenoid



coil 1: $\phi_{B_1} = \phi_{B11} + \phi_{B12}$

coil 2: $\phi_{B_2} = \phi_{B21} + \phi_{B22}$

ϕ_{B21} = flux thru coil 2 due to B_1 = $\mu_0 n_1 n_2 l A\, i_1$

ϕ_{B12} = flux thru coil 1 due to B_2 = $\mu_0 n_1 n_2 l A\, i_2$

ϕ_{B11} = self-linkages (coil 1) = $\mu_0 n_1^2 l A\, i_1$

ϕ_{B22} = self-linkages (coil 2) = $\mu_0 n_2^2 l A\, i_2$

So Faraday's law $\Rightarrow$

$$\mathcal{E}_1 = -L_1 \frac{di_1}{dt} - M \frac{di_2}{dt}$$

$$\mathcal{E}_2 = -L_2 \frac{di_2}{dt} - M \frac{di_1}{dt}$$

$M = \sqrt{L_1 L_2}$

L_1 = self-inductance of coil 1 = $\frac{\phi_{B11}}{i_1} = \mu_0 n_1^2 l A$

L_2 = self-inductance of coil 2 = $\frac{\phi_{B22}}{i_2} = \mu_0 n_2^2 l A$

M = mutual inductance = $\frac{\phi_{B12}}{i_2} = \frac{\phi_{B21}}{i_1} = \mu_0 n_1 n_2 l A$

-VII. c.) Magnetic Energy: Work required to move charge through inductor

$$dW = -\mathcal{E}\,dq = dq\,\frac{d\phi_B}{dt}$$

But $dq = i\,dt$ & $\frac{d\phi_B}{dt} = \frac{d\phi_B}{di}\frac{di}{dt} = L\frac{di}{dt}$.

Hence $\quad dW = i\,dt\,L\frac{di}{dt} = L\,i\,di$

This is stored in the magnetic field of the inductor — it is the increase in Potential Energy stored in the magnetic field:

$$dU_B = dW = L\,i\,di$$

$$\Rightarrow \quad \boxed{U_B = \frac{1}{2}Li^2 = \frac{1}{2}i\,\phi_B}$$

1) Applied to a solenoid: $L = \mu_0 n^2 A l \Rightarrow$

$$U_B = \frac{1}{2}\mu_0 n^2 i^2 (Al) \quad \text{but } B = \mu_0 n i$$

$$\Rightarrow \quad \boxed{U_B = \frac{1}{2\mu_0}B^2(Al) = \frac{1}{2\mu_0}\int_{\substack{\text{All}\\\text{Space}}} \vec{B}\cdot\vec{B}\,dV}$$

2) Energy density stored in magnetic field

$$\boxed{u_B = \frac{U_B}{Al} = \frac{1}{2\mu_0}\vec{B}\cdot\vec{B}}\;;\; U_B = \int_{\substack{\text{All}\\\text{Space}}} u_B\,dV$$

VII. C) 3) If a magnetic material is present

$$\boxed{U_B = \frac{1}{2}\int_{\substack{\text{All}\\\text{Space}}} \vec{B}\cdot\vec{H}\,dV \qquad u_B = \frac{1}{2}\vec{B}\cdot\vec{H}}$$

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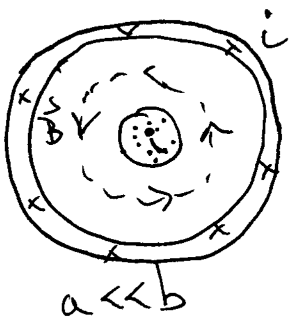
4) Example: Coaxial Cable: What is the energy stored for a length l of cable.



inner conductor radius = a
outer conductor inner radius = b

Ampere's Law

$$\oint_C \vec{B}\cdot d\vec{s} = \mu_0 i_{enclosed}$$

$$B\,2\pi r = \mu_0 i$$

$$B(r) = \frac{\mu_0 i}{2\pi r}$$

So energy stored in magnetic field $(a \le r \le b)$

$$u_B = \frac{1}{2\mu_0}B^2 = \frac{\mu_0 i^2}{8\pi^2 r^2}$$

Assume center conductor is small radius and outer conductor thin $\Rightarrow$

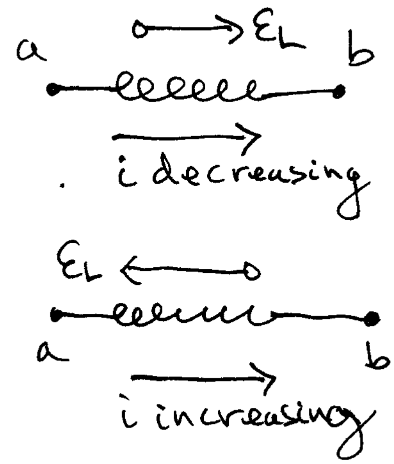
$$U_B = \frac{1}{2}\int_{r=a}^{b} \frac{1}{\mu_0}B^2\,dV = \frac{\mu_0 i^2 l}{4\pi}\ln\left(\frac{b}{a}\right)$$

VII.C.4) Recalling that $U_B = \frac{1}{2} L i^2$ we can determine the inductance of the cable from U_B above

$$\boxed{L = \frac{\mu_0 l}{2\pi} \ln\left(\frac{b}{a}\right)}$$ which agrees with $L = \frac{\Phi_B}{i}$

D) LR-Circuits

1) Potential Difference across inductor (Lenz' Law)



$V_b > V_a$

$V_a > V_b$

$V_b - V_a = -L \frac{di}{dt}$ in both cases

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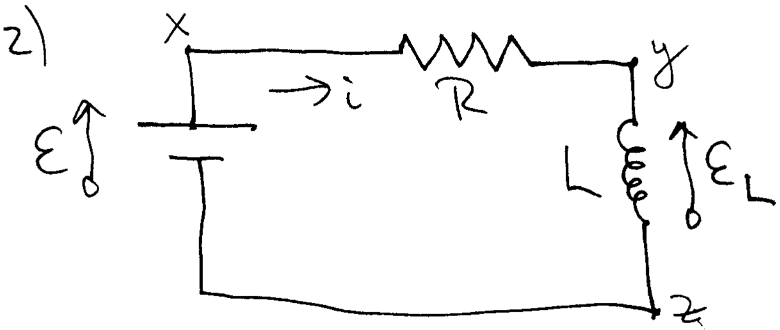
2)



Current Build Up

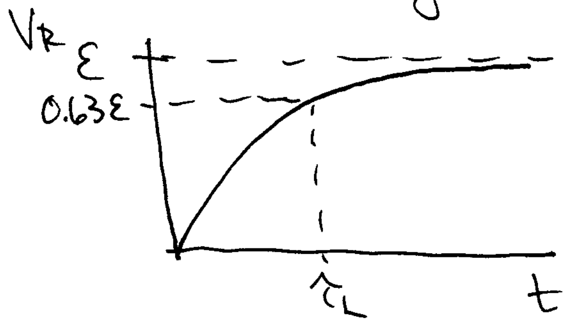
Loop Rule: $\mathcal{E} = iR + L\frac{di}{dt}$

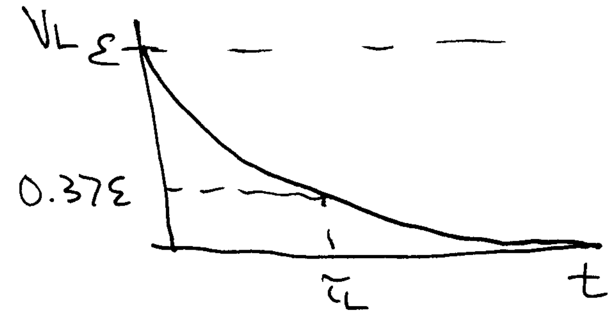
DE for i $\Rightarrow$ $i(t) = \frac{\mathcal{E}}{R}\left(1 - e^{-t/\tau_L}\right)$

-VII.D) 2) with $\tau_L \equiv L/R$ = inductive time constant

So $V_R = V_x - V_y = iR = \mathcal{E}(1 - e^{-t/\tau_L})$

$V_L = V_y - V_z = -\mathcal{E}_L = L\frac{di}{dt} = \mathcal{E}e^{-t/\tau_L}$





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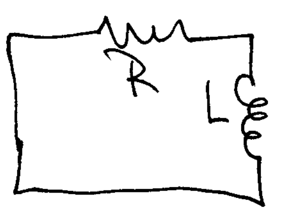
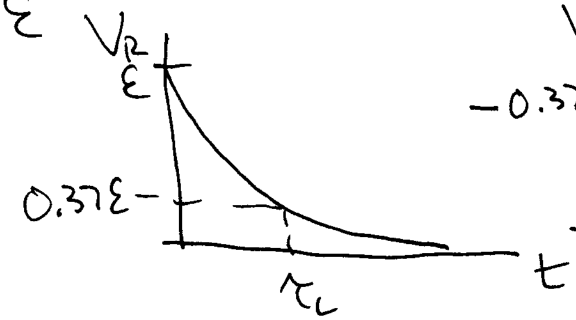
3) Current Decay

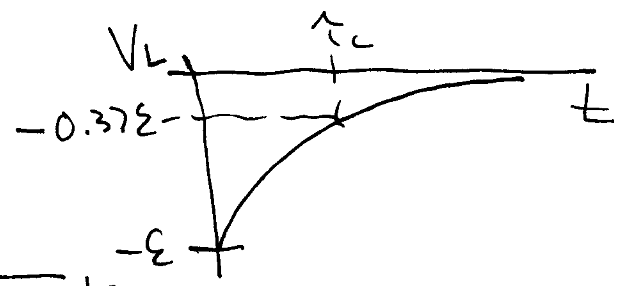


Loop Rule: $L\frac{di}{dt} + iR = 0$

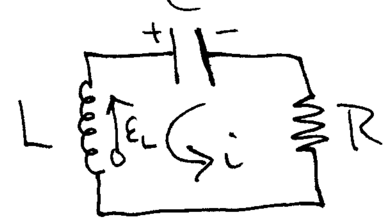
$\Rightarrow \boxed{i(t) = i_0 e^{-t/\tau_L}}$

Let $i_0 = \mathcal{E}$





VII. E.) LCR-Circuit: Initially charged capacitor



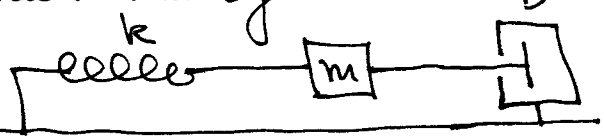
Loop Rule:

$$L\frac{d^2q}{dt^2} + R\frac{dq}{dt} + \frac{1}{C}q = 0$$

with initial conditions $q(0) = q_m$

$$\frac{dq(0)}{dt} = 0.$$

(Mechanical Analog



$F = -kx - bv$

Newton's 2nd Law $\vec{F} = m\vec{a} \Rightarrow$

with i.c. $x(0) = x_m$

$\frac{dx(0)}{dt} = 0$

$$m\frac{d^2x}{dt^2} + b\frac{dx}{dt} + kx = 0 \quad)$$

For $4\frac{L}{C} > R^2$ $\quad q(t) = q_0 e^{-\frac{Rt}{2L}} \cos(\omega' t + \phi)$

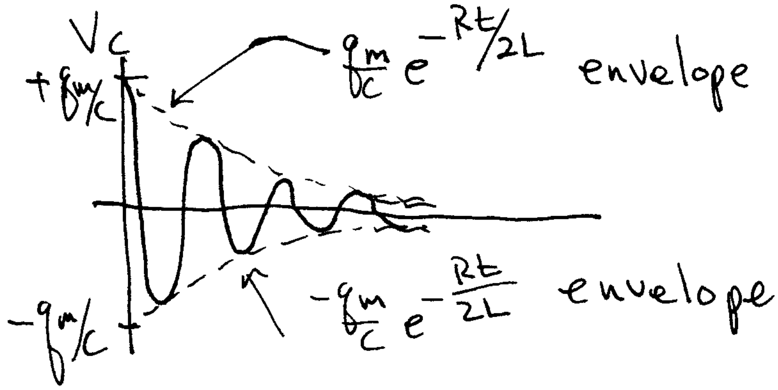
i.c. $\Rightarrow$ $\tan\phi = \dfrac{-1}{\sqrt{\frac{4L}{RC^2} - 1}}$; $q_0 = q_m \dfrac{1}{\sqrt{LC}\,\omega'}$

1) Energy Loss:

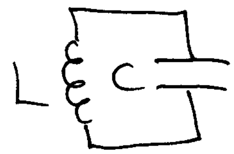
$$U = U_E + U_B = \frac{1}{2}\frac{q^2}{C} + \frac{1}{2}Li^2$$

$$\frac{dU}{dt} = -i^2R \quad \text{Joule heating}$$

2)



VII. E. 3) $R \to 0$



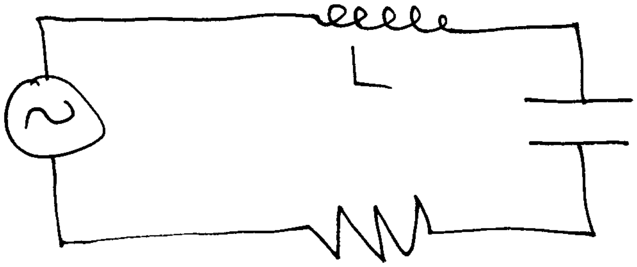
$$\frac{dU}{dt} = 0$$

$$U = \text{constant} = U_E + U_B = \frac{1}{2}\, q_m^2/C$$

energy oscillates between being stored in electric field and the magnetic field as the current oscillates sinusoidally CW then CCW in circuit.

F.) Driven LCR-Circuit — Resonance

$\mathcal{E} = \mathcal{E}_m \sin\omega t$



Loop Equation $\Rightarrow$ $$L\frac{d^2q}{dt^2} + R\frac{dq}{dt} + \frac{1}{C}q = \mathcal{E}_m \sin\omega t$$

(Mechanical Analog

$F_{ext} = F_m \sin\omega t$

)

$$q(t) = q_c(t) + q_p(t)$$

complimentary solution = transient solution above

particular solution = steady state solution

VII. F) Ansatz: $q_p(t) = \frac{i_m}{\omega}\sin(\omega t - \delta)$

Conventional $\delta \equiv \phi + \frac{\pi}{2}$ so

$$i = \dot{q}_p(t) = i_m \sin(\omega t - \phi)$$

Loop equation $\Rightarrow$ $\tan\delta = \dfrac{\frac{R}{L}\omega}{\omega_0^2 - \omega^2}$

or 1) $\boxed{\tan\phi = \dfrac{\omega^2 - \omega_0^2}{\frac{R}{L}\omega}}$

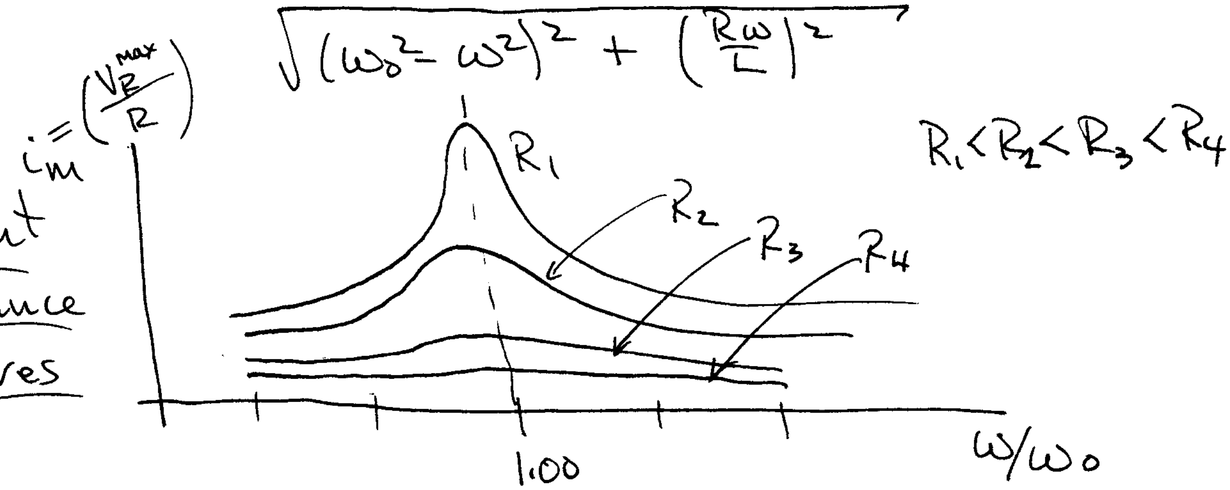
where

$$\omega_0^2 \equiv \frac{1}{LC} \quad \text{(natural frequency of LC circuit)}$$

2) $i_m = \dfrac{\mathcal{E}_m}{R}\sec\phi$

$$= \frac{\omega \cdot \mathcal{E}_m / L}{\sqrt{(\omega_0^2 - \omega^2)^2 + \left(\frac{R\omega}{L}\right)^2}}$$

Current Resonance Curves



$$\omega_R = \omega_0 = \frac{1}{\sqrt{LC}} \quad \left(\text{Current } (V_R) \text{ Resonance}\right)$$

VII. F. 3) Voltage drop across capacitor $= V_c = \frac{q_p}{C}$

$$V_{c_{max}} = \frac{i_m}{\omega C}$$ has resonance at charge q_p resonance at

frequency: $\omega_{q_R} = \sqrt{\omega_0^2 - \frac{1}{2}\left(\frac{R}{L}\right)^2} < \omega_0$

4) Voltage drop across inductor $= V_L = L\frac{di}{dt}$

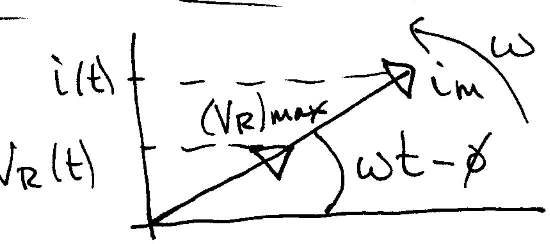
$$V_{L_{max}} = L\omega i_m$$ has resonance at

frequency: $$\omega_{L_R} = \frac{\omega_0}{\sqrt{1 - \frac{1}{2}\left(\frac{R^2C}{L}\right)}} > \omega_0$$

G) <u>Phasor</u> <u>Diagrams</u>: $i(t) = i_m \sin(\omega t - \phi)$ through circuit element

1) <u>Resistive Load</u>: $V_R = i_m R \sin(\omega t - \phi)$

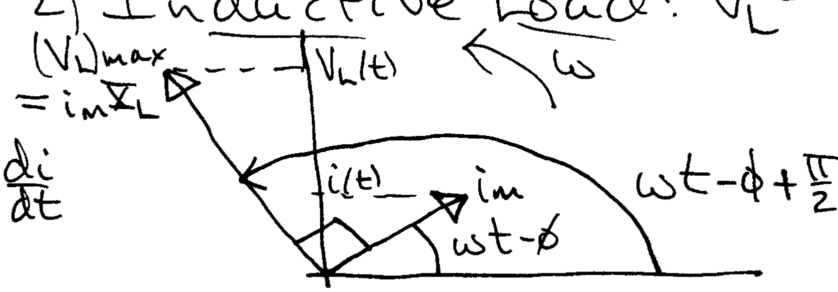
$V_R = iR$



V_R & i are in phase

2) <u>Inductive Load</u>: $V_L = i_m X_L \sin(\omega t - \phi + \frac{\pi}{2})$

$V_L = L\frac{di}{dt}$

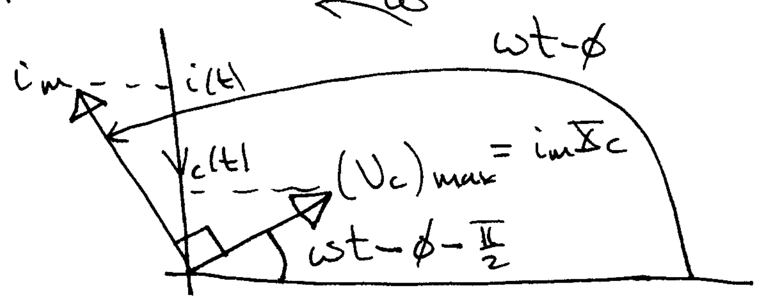


a) ELI

b) $X_L \equiv \omega L$ = inductive reactance

VIII. G. 3) <u>Capacitive Load</u>: $V_C = i_m X_C \sin(\omega t - \phi - \frac{\pi}{2})$

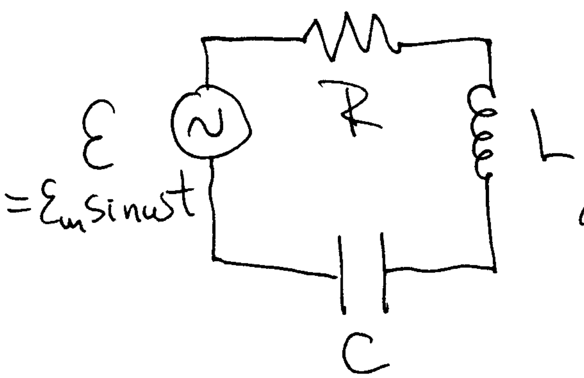
$V_C = \frac{q}{C} = \frac{\int i\,dt}{C}$



a) ICE

b) $X_C \equiv \frac{1}{\omega C}$ = capacitive reactance

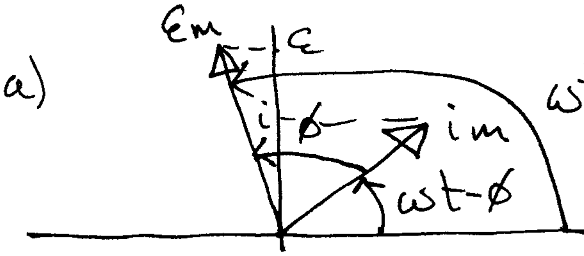
4) <u>LCR-Circuit</u>:



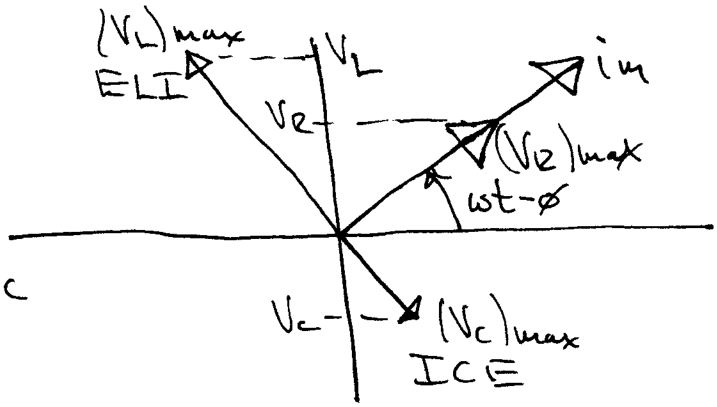
$\mathcal{E} = \mathcal{E}_m \sin \omega t$

Ansatz: $i = i_m \sin(\omega t - \phi)$

a)



b) RLC Components:



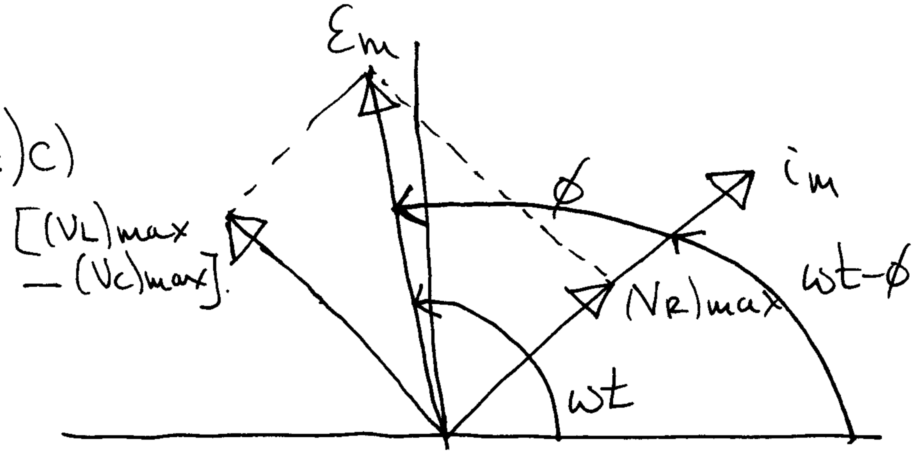
loop Rule:

$\mathcal{E} = V_R + V_L + V_C$

(scalar sum)

c) <u>Vector</u> sum of $(V_R)_{max}$, $(V_L)_{max}$, $(V_C)_{max}$ <u>phasors</u> <u>equals</u> <u>$\mathcal{E}_m$</u> <u>phasor</u>

- VII. G 4) c)



Since $V_{R,C,L}$ phasors are at right angles $\Rightarrow$

$$\mathcal{E}_m = \sqrt{[(V_R)_{max}]^2 + [(V_L)_{max} - (V_C)_{max}]^2}$$

$$= i_m \sqrt{R^2 + (X_L - X_C)^2}$$

$$\boxed{\begin{gathered} \Rightarrow \\ i_m = \mathcal{E}_m / Z \\ Z \equiv \sqrt{R^2 + (X_L - X_C)^2} = \text{impedance} \end{gathered}}$$

$$\tan\phi = \frac{(V_L)_{max} - (V_C)_{max}}{(V_R)_{max}} = \boxed{\frac{X_L - X_C}{R} = \tan\phi}$$

1) Resonance for the current occurs at $\phi = 0$
$\Rightarrow X_L = X_C \Rightarrow \omega = \omega_0 = \frac{1}{\sqrt{LC}}$,
So $(V_L)_{max}$ and $(V_C)_{max}$ are equal and opposite to give i_m in phase with $\mathcal{E}_m$.

VII. H) Power in AC circuits:

1) Power dissipation in Resistor:

$$P = i^2 R = i_m^2 R \sin^2(\omega t - \phi)$$

Average power dissipation

$$\overline{P} \equiv \frac{1}{N\tau}\int_0^{N\tau} P(t)\,dt$$

$$= \frac{1}{2} i_m^2 R = i_{rms}^2 R$$

with root mean square current

$$i_{rms} = \frac{i_m}{\sqrt{2}} \equiv \sqrt{\overline{(i)^2}}\ .$$

2) Power supplied to circuit by source of emf

$$P = \frac{dW}{dt} = \mathcal{E} i = \mathcal{E}_m i_m \sin\omega t \sin(\omega t - \phi)$$

Average power supplied

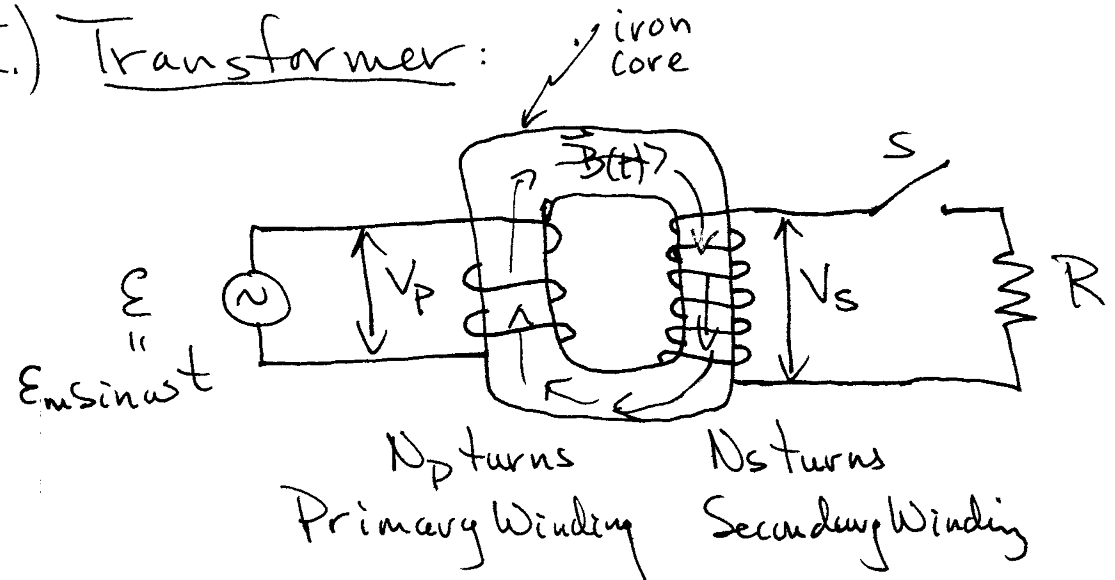
$$\boxed{\begin{aligned}\overline{P} &= \tfrac{1}{2}\mathcal{E}_m i_m \cos\phi \\ &= \mathcal{E}_{rms}\, i_{rms} \cos\phi\end{aligned}}$$

VIII.H.2) $\cos\phi \equiv$ power factor

$$= \frac{R}{Z} = \frac{R}{\sqrt{R^2 + (X_L - X_C)^2}}$$

If resistive load $X_L = X_C \Rightarrow \cos\phi = 1$
power delivered to circuit is maximum
$\bar{P} = \mathcal{E}_{rms} i_{rms}$ ($Z = R$ resistive load)

I.) Transformer:



1) $$\left(\frac{d\phi_B}{dt}\right)_{\substack{\text{primary}\\ \text{per turn}}} = \left(\frac{d\phi_B}{dt}\right)_{\substack{\text{secondary}\\ \text{per turn}}}$$

$$\Rightarrow \frac{V_p}{N_p} = \frac{V_s}{N_s} \Rightarrow \boxed{V_s = V_p\left(\frac{N_s}{N_p}\right)}$$

2) Energy Conservation: $i_p V_p = i_s V_s$ (Resistive load)

$$\Rightarrow \boxed{i_s = i_p\left(\frac{N_p}{N_s}\right)}$$
