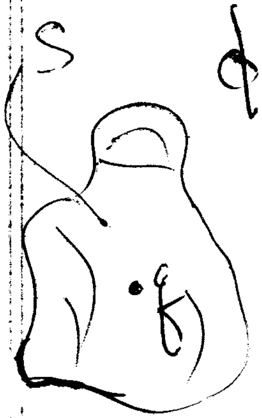
II B.2.) So the solid ∡ = ∮ flux of $\frac{\vec{r}}{r^3} = \frac{\hat{r}}{r^2}$.

<u>example</u>: What is the flux of $\vec{E}$ through a surface S enclosing a point charge q?



$$\phi = \int_S \vec{E} \cdot d\vec{S} = \int_S \frac{q}{4\pi\epsilon_0} \frac{\hat{r}}{r^2} \cdot d\vec{S}$$

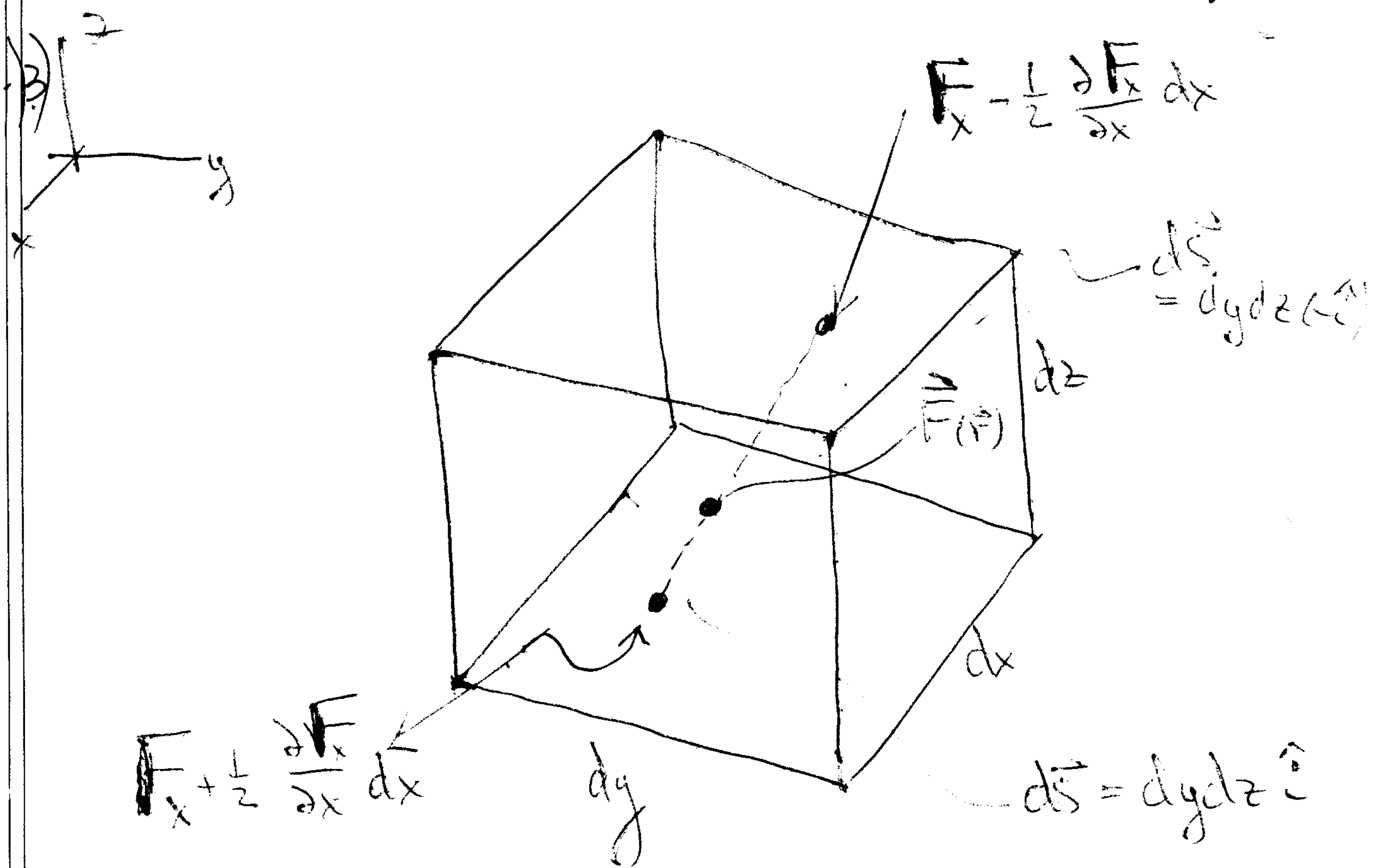
$$= \frac{q}{4\pi\epsilon_0} \underset{4\pi}{\underset{\parallel}{\Omega_S}} = \frac{q}{\epsilon_0}$$

Example of Gauss's Law.

II.B.3.) Next, we will relate the divergence of a vector field and the flux of a vector.

Consider the vector field $\vec{F}(\vec{r})$ (eg. the velocity field of a fluid) and let $\underline{\vec{F}(\vec{r})}$ be the <u>value</u> of the <u>vector</u> at the center of an <u>infinitesimal element of volume</u> with sides dx, dy, dz and center at $\vec{r}$.

II.B.3)



Consider the 2 faces ⊥ to the x-axis they have the area $dydz$. On the forward face the x-component of $\vec{F}$ at the center of the face is given by

$$F_x(\vec{r}) + \frac{1}{2}dx\,\frac{\partial F_x(\vec{r})}{\partial x}$$

[Recall Taylor's Thm. $V(\vec{r}+\Delta\vec{r}) = V(\vec{r}) + dV(\vec{r})$

$$= V(\vec{r}) + \vec{\nabla}V(\vec{r})\cdot\Delta\vec{r}$$

here $\Delta\vec{r} = \frac{1}{2}\Delta x\,\hat{i}$]

II.B.)3.) We can consider this to be the value of the x-component over the whole face as the face becomes small.

On the near face we find the x-component is

$$F_x - \frac{1}{2} dx \frac{\partial F_x}{\partial x} .$$

Now recall the flux of $\vec{F}$ through the forward surface is

$$\phi_{forward} = \vec{F} \cdot d\vec{S} \Big|_{\substack{\text{on forward} \\ \text{surface}}} = \left(F_x + \frac{1}{2} \frac{\partial F_x}{\partial x} dx\right) dy\,dz$$

The near flux is

$$\phi_{near} = \vec{F} \cdot d\vec{S} \Big|_{\substack{\text{on near} \\ \text{surface}}} = -\left(F_x - \frac{1}{2} \frac{\partial F_x}{\partial x} dx\right) dy\,dz$$

So the net flux in the x-direction is the sum

II.B.)3.) $\phi_x = \phi_{forward} + \phi_{rear}$

$$= \left(F_x + \frac{1}{2}\frac{\partial F_x}{\partial x}dx\right)dydz$$

$$- \left(F_x - \frac{1}{2}\frac{\partial F_x}{\partial x}dx\right)dydz$$

$$= \frac{\partial F_x}{\partial x}\,dxdydz$$

Similarly the net flux in the y & z directions is just

$$\phi_y = \frac{\partial F_y}{\partial y}\,dxdydz$$

$$\phi_z = \frac{\partial F_z}{\partial z}\,dxdydz$$

Thus the total flux ϕ over the infinitesimal surface element is

$$\phi = \left(\frac{\partial F_x}{\partial x} + \frac{\partial F_y}{\partial y} + \frac{\partial F_z}{\partial z}\right)dxdydz$$

$$= \underbrace{\operatorname{div}\vec{F}}\,dxdydz$$

---

use $(\vec{\nabla}\cdot\vec{F})$ for divergence

II.B.)3.) Note:

Thus we have arrived at our coordinate independent definition of the divergence

$\text{div}\vec{F} = \vec{\nabla}\cdot\vec{F}$ is the amount of flux per unit volume in the limit the volume $\to 0$.

Thus

$$\vec{\nabla}\cdot\vec{F} = \text{div}\vec{F} = \lim_{\Delta V\to 0} \frac{1}{\Delta V}\iint_S \vec{F}\cdot d\vec{S}$$

$$= \lim_{\Delta V\to 0}\frac{\Phi}{\Delta V}$$

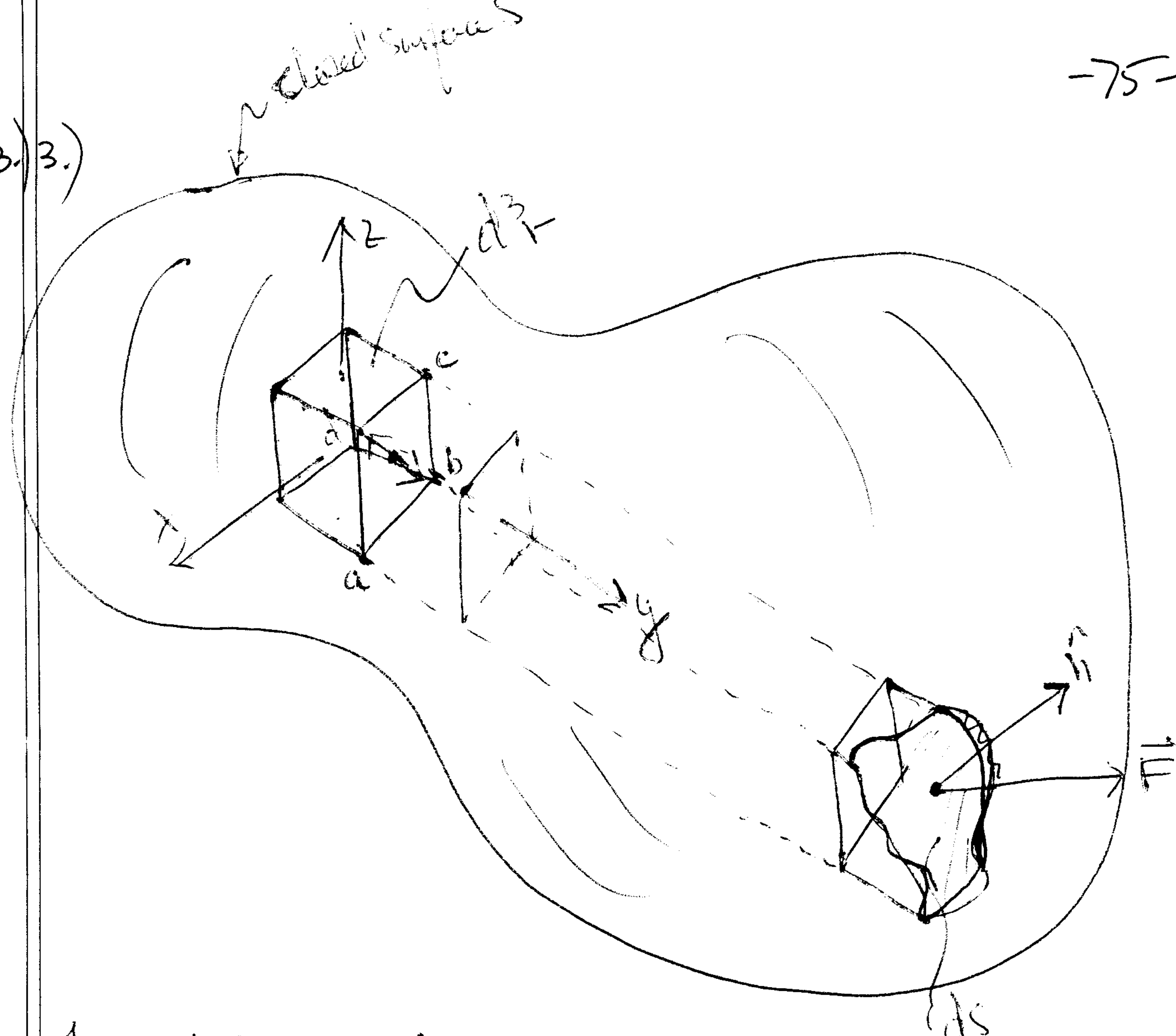
ex. 1. Gauss's Theorem

Consider a closed surface S in a vector field $\vec{F}$. The total flux through the surface was defined by

$$\Phi = \iint_S \vec{F}\cdot d\vec{S}$$

Next consider an infinitesimal vol. d^3r inside S i.e. cube with edges $\parallel$ to the x-y-z axes.

II.B.) 3.)



The total flux diverging from d^3r is just

$$d\phi = \mathrm{div}\,\vec{F}\,d^3r. = \vec{\nabla}\cdot\vec{F}\,d^3r$$

For the face abcd the positive y-comp of $\vec{F}$ & the normal to the face are taken outward — thus for the cube next to it $\vec{F}$ & its outward normal are in opposite directions - since it is the same face the contributions to the flux

II.B.3. from each cancels. We continue this until the last volume element whose one face is part of S. This alone will make a contribution to the flux through S. We continue this process throughout the volume V enclosed by S. This is we have integrated $\vec{\nabla}\cdot\vec{F}\,d^3r$ throughout the enclosed volume V to obtain the total flux through S. Thus

$$\phi = \iint_S \vec{F}\cdot d\vec{S} = \iiint_V \operatorname{div}\vec{F}\,d^3r$$

$$= \iiint_V \vec{\nabla}\cdot\vec{F}\,\underbrace{d^3r}_{=d\tau}$$

This is Gauss's Divergence Theorem

See Schaum's p. 117 for a more mathematically satisfying proof.

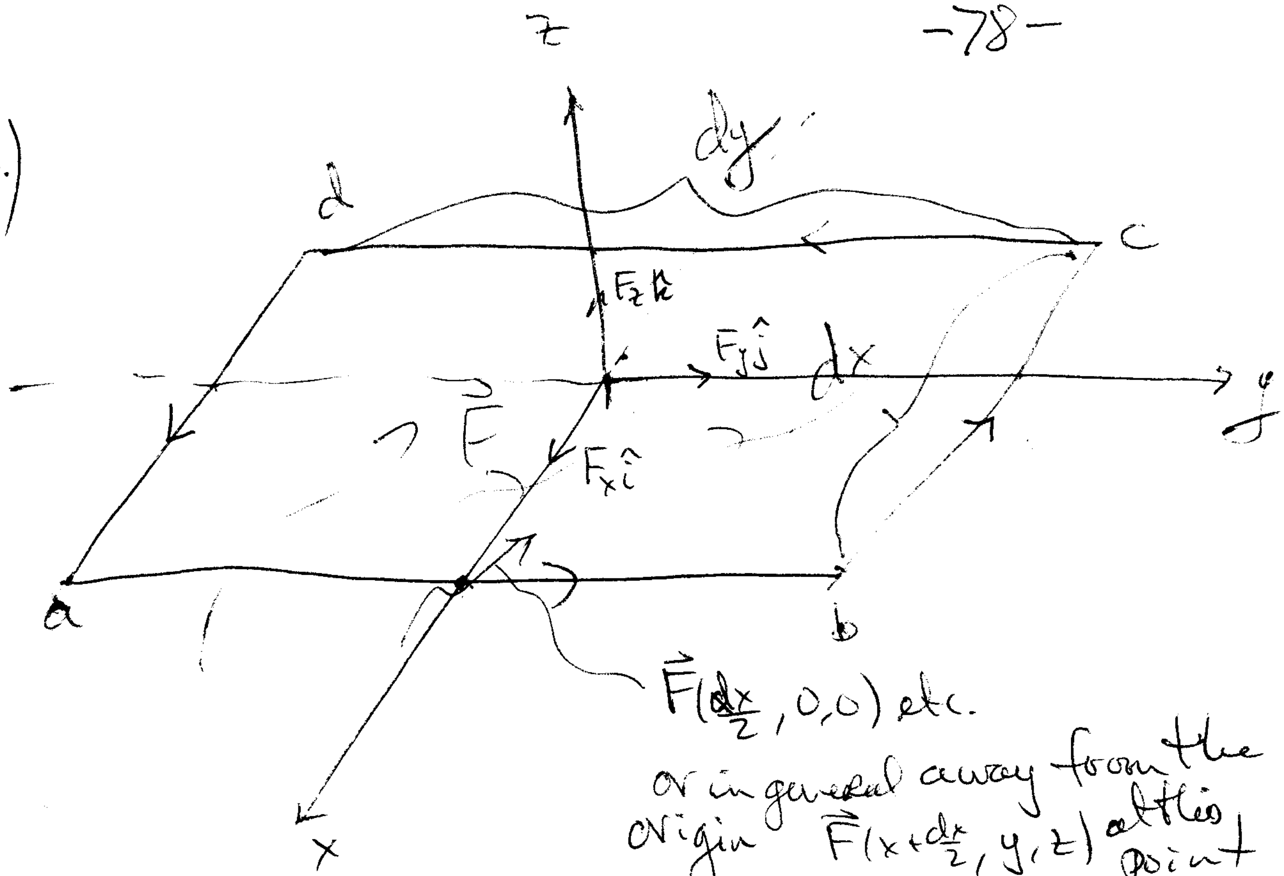
II.B)3.) So in words Gauss's Theorem: The integral of the divergence of any vector over a volume V bounded by a closed surface S is equal to the flux of the vector through the surface S.

$$\iint_S \vec{F} \cdot d\vec{S} = \iiint_V \vec{\nabla} \cdot \vec{F} \, d^3r$$

II.B) 4) We can next relate the flux of the curl of a vector through an open surface to the line integral of the vector around the boundary of the surface - this is known as Stokes's theorem

Consider the line integral of $\vec{F}$ along the curve ~~ABCD~~ abcd in the x-y plane

II.B.4.)



The value of the vector field is $\vec{F}$ in the center of the rectangle, the value of $\vec{F}$ at the middle of the side may be taken as the value of $\vec{F}$ along the whole side

So along side

ab
(|| to y-axis)
y component

$$F_y + \frac{\partial F_y}{\partial x}\frac{dx}{2}$$

(cd) y component

$$F_y - \frac{\partial F_y}{\partial x}\frac{1}{2}dx$$

I.B.4.) (ad) x comp. $\quad F_x - \frac{\partial F_x}{\partial y}\frac{1}{2}dy$

(bc) x comp. $\quad F_x + \frac{1}{2}\frac{\partial F_x}{\partial y}dy$.

The line integral around abcd is then

$$\oint_C \vec{F}\cdot d\vec{r} = \left(F_y + \frac{1}{2}\frac{\partial F_y}{\partial x}dx\right)dy - \left(F_y - \frac{1}{2}\frac{\partial F_y}{\partial x}dx\right)dy + \left(F_x - \frac{1}{2}\frac{\partial F_x}{\partial y}dy\right)dx - \left(F_x + \frac{1}{2}\frac{\partial F_x}{\partial y}dy\right)dx$$

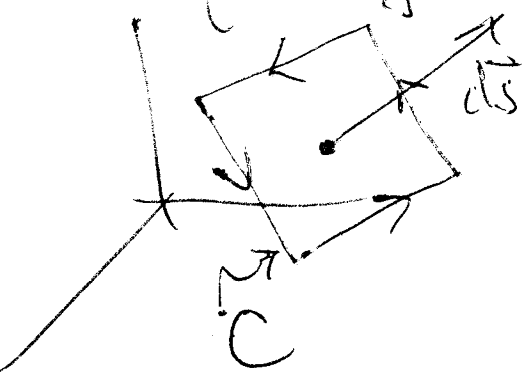
$$= \left[\frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y}\right]dx\,dy$$

$$= \vec{\nabla}\times\vec{F}\cdot d\vec{S} \text{ where}$$

RH rule convention. $\quad d\vec{S} = dx\,dy\,\hat{k}$

We can do the same thing for loops in the other 2 planes. In general for any plane area we find

$$\oint_C \vec{F}\cdot d\vec{r} = \vec{\nabla}\times\vec{F}\cdot d\vec{S}$$



II.B.4) Thus we arrive at our coordinate independent definition of the curl.

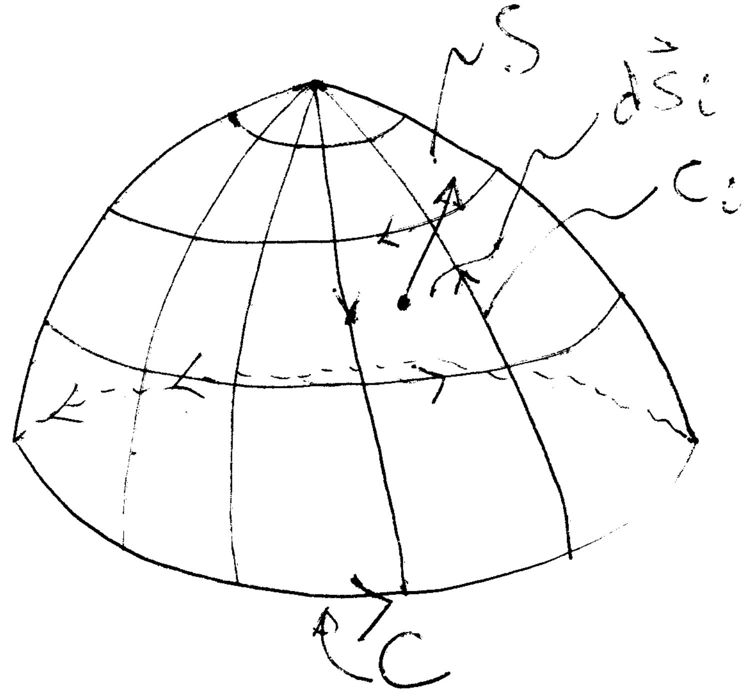
The component of $\vec{\nabla} \times \vec{F}$ in the direction of the unit vector $\hat{n}$ is the limit of a line integral per unit area, as the enclosed area goes to zero, this area having normal ∥ to $\hat{n}$. That is,

$$\hat{n} \cdot \left(\vec{\nabla} \times \vec{F} \right) = \lim_{S \to 0} \frac{1}{S} \oint_C \vec{F} \cdot d\vec{l}$$

where curve C bounds S & is ⊥ to $\hat{n}$.

IV. Stokes's Theorem

For any open surface we can divide the surface into small plane surfaces



II.B.4.) Then

$$\oint_{C_i} \vec{F}\cdot d\vec{l} = \overrightarrow{\text{curl} F}\cdot d\vec{S}_i = (\vec{\nabla}\times\vec{F})\cdot d\vec{S}_i$$

we now add up the areas

$$\sum_{i=1}^{n} \oint_{C_i} \vec{F}\cdot d\vec{l} = \sum_{i=1}^{n} \underbrace{\overrightarrow{\text{curl} F}}_{=(\vec{\nabla}\times\vec{F})}\cdot d\vec{S}_i$$

As $dS_i \to 0$ the line integrals along common edges cancel leaving only the line integral around the boundary C of S

$$\boxed{\oint_C \vec{F}\cdot d\vec{l} = \iint_S (\vec{\nabla}\times\vec{F})\cdot d\vec{S}}$$

This is Stokes Theorem: the surface integral over an open surface of the curl of a vector field is equal to the line integral of the vector along the curve that is the boundary of the surface.

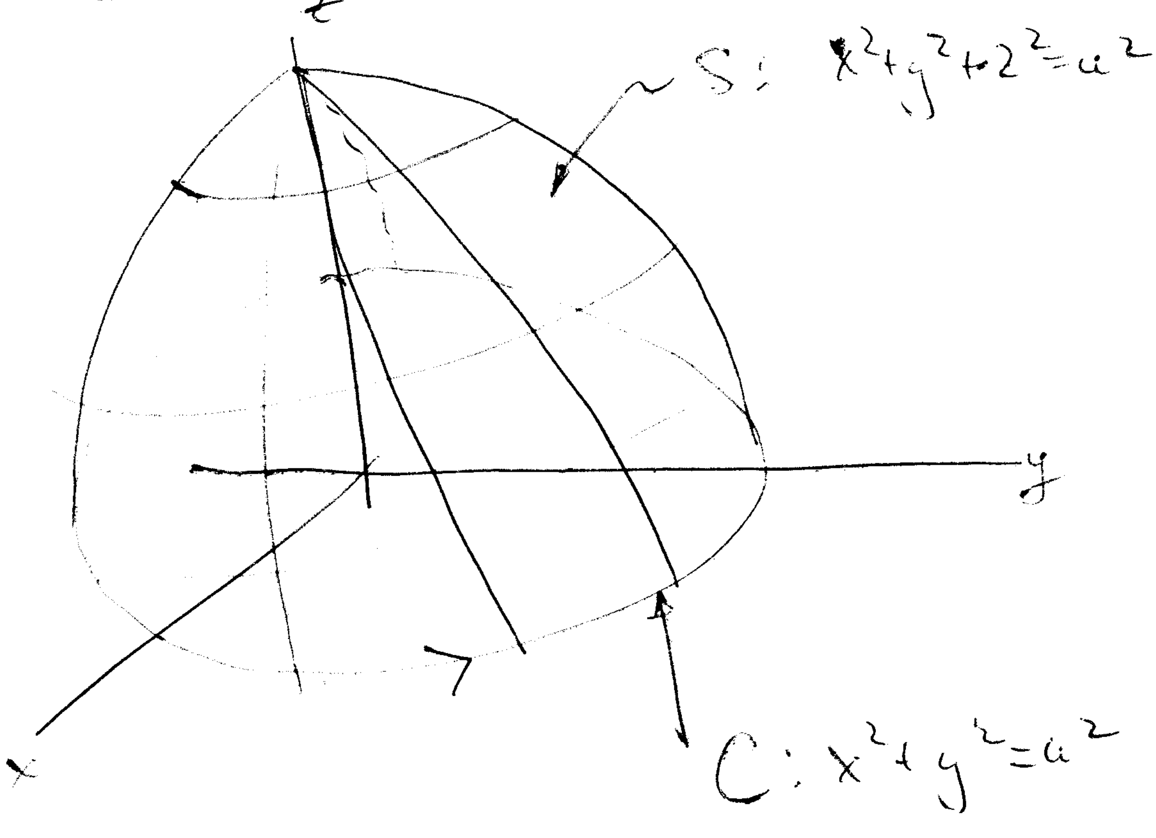
II.B.4 ex. Evaluate $\iint_S \text{curl}\,\vec{F}\cdot d\vec{S}$ where

$\vec{F} = -y\hat{i} + x\hat{j} + z\hat{k}$ and S is the upper half of the sphere $x^2+y^2+z^2=a^2$. insert page -83-

By Stokes's theorem we can evaluate this also as a line integral

$$\iint_S \text{curl}\,\vec{F}\cdot d\vec{S} = \oint_C \vec{F}\cdot d\vec{l}$$

where C is the circle in the x-y plane bounding the hemisphere $x^2+y^2=a^2$.



II.B.4.) The curl of $\vec{F}$ is simply

$$\vec{\nabla}\times\vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial_x & \partial_y & \partial_z \\ F_x & F_y & F_z \end{vmatrix} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial_x & \partial_y & \partial_z \\ -y & x & z \end{vmatrix}$$

$$= 2\hat{k}$$

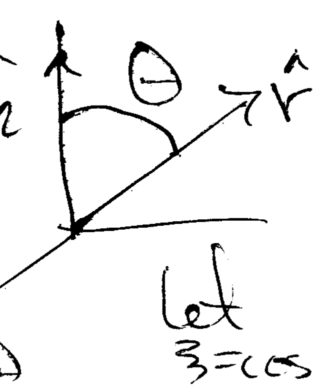
The element of surface area in the Northern hemisphere is, for $r = a$,

$$d\vec{S} = r^2 \sin\theta \, d\varphi \, d\theta \; \hat{r}$$
$$= a^2 \sin\theta \, d\theta \, d\varphi \; \hat{r}$$

Now $(\vec{\nabla}\times\vec{F})\cdot d\vec{S}$ on the surface of the hemisphere is

$$(\vec{\nabla}\times\vec{F})\cdot d\vec{S} = 2a^2 \sin\theta \, d\theta \, d\varphi \; \underbrace{\hat{k}\cdot\hat{r}}_{=\cos\theta}$$

$$= 2a^2 \cos\theta \underbrace{\sin\theta \, d\theta}_{=d(\cos\theta)} d\varphi$$



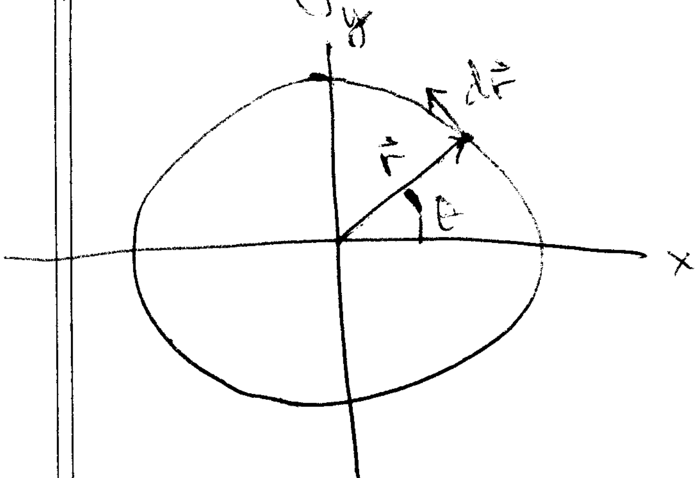
So hemisphere

$$\boxed{\iint_S \vec{\nabla}\times\vec{E}\cdot d\vec{S} = 2\pi a^2} = \int_{\varphi=0}^{2\pi} d\varphi \int_{\theta=0}^{\pi/2} 2a^2 \cos\theta \sin\theta \, d\theta \quad \text{let } \xi = \cos\theta$$

$$= 4\pi a^2 \int_{\xi=0}^{1} \xi \, d\xi = \underline{2\pi a^2}$$

φ integral $= 2\pi$

II.B.4) Now along the circle we have



$$x = a\cos\theta$$
$$y = a\sin\theta$$
$$d\vec{r} = dx\,\hat{i} + dy\,\hat{j}$$
$$dx = -a\sin\theta\,d\theta$$
$$dy = a\cos\theta\,d\theta$$

So
$$\oint_C \vec{F}\cdot d\vec{r} = \oint_C (-y\,dx + x\,dy)$$
$$= \int_{\theta=0}^{2\pi} [-a\sin\theta][-a\sin\theta\,d\theta] + (a\cos\theta)(a\cos\theta\,d\theta)$$
$$= a^2\int_{\theta=0}^{2\pi} (\sin^2\theta + \cos^2\theta)\,d\theta = a^2\int_{\theta=0}^{2\pi} d\theta$$
$$= 2\pi a^2$$

So
$$\boxed{\iint_S \vec{\nabla}\times\vec{F}\cdot d\vec{S} = \oint_C \vec{F}\cdot d\vec{r} = 2\pi a^2}$$

in agreement with the direct calculation !!!!

This ends Vector Calculus — See the Appendix for more ∇ identities & Curvilinear Coord.