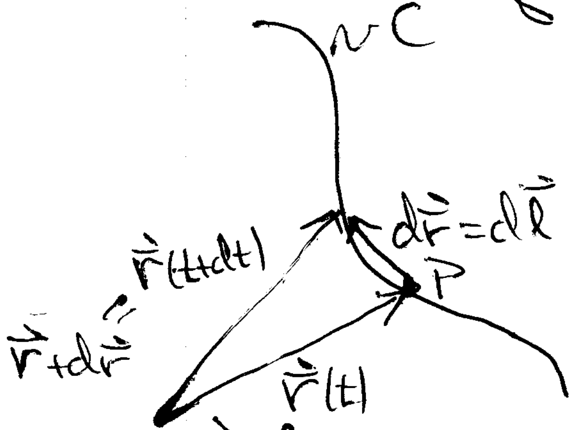
II.)B.) Vector Integration:

1) The Line Integral:

Let C be any curve and let $\vec{r}(t)$ trace out the curve, then $d\vec{r}$ ($d\vec{\ell}$) is an element of arc along the curve



At any point P, $d\vec{r}$ is in the direction of the unit tangent at P and oriented towards increasing arc length.

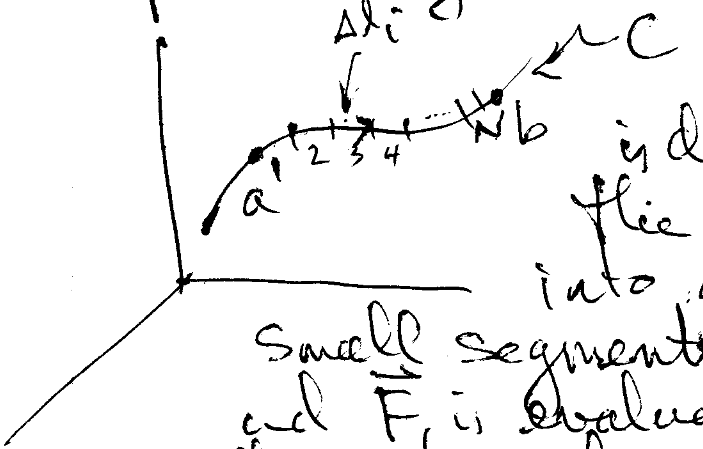
Then we can define the following integrals

$$\int_{a \text{ along } C}^{b} f(x,y,z)\, d\vec{r} \;;\; \int_{C\,a}^{b} \vec{F}\cdot d\vec{r} \;;\; \int_{C\,a}^{b} \vec{F}\times d\vec{r}.$$

First consider the second integral

$$\int_{C\,a}^{b} \vec{F}\cdot d\vec{\ell}$$

II.B.1) a and b are the initial and final points along curve C



The Riemann integral is defined by dividing the line between a & b into a large number of small segments of length $\vec{\Delta l}_i$ and $\vec{F}$ is evaluated at a point in the interval; call it $\vec{F}_i$.

Then

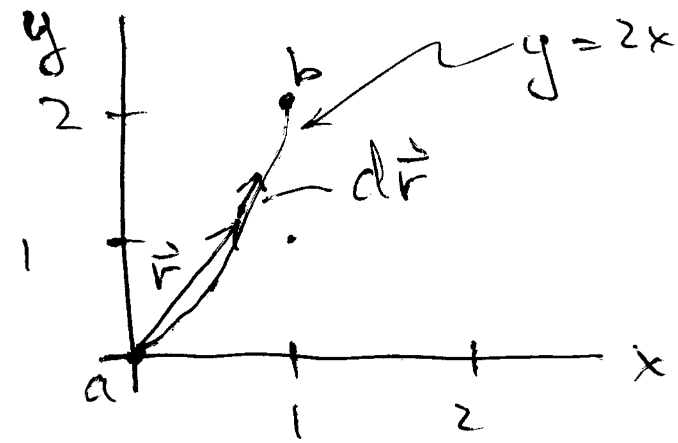
$$\int_{a \atop C}^{b} \vec{F} \cdot d\vec{l} \equiv \lim_{\substack{N\to\infty \\ |\Delta l_i|\to 0}} \sum_{i=1}^{N} \vec{F}_i \cdot \vec{\Delta l}_i$$

Note: the line integral depends on endpoints a and b <u>and</u> on the curve C.

If C is a closed curve we use the notation

$$\oint_C \vec{F} \cdot d\vec{l}$$

II.B.1.) Ex. If $\vec{F} = 3xy\,\hat{i} - y^2\hat{j}$, evaluate $\int_{C\,a}^{b} \vec{F}\cdot d\vec{\ell}$, where C is the curve in the x-y plane $y = 2x^2$ from $a = (0,0)$ to $b = (1,2)$



So $d\vec{r} = dx\,\hat{i} + dy\,\hat{j}$ in general,

but along the curve C $y = 2x^2 \Rightarrow$ use x to param. curve

$dy = 4x\,dx$. Thus $= (6x^3\hat{i} - 4x^4\hat{j})\cdot(\hat{i} + 4x\hat{j})\,dx$

$$\int_{C\,a}^{b} \vec{F}\cdot d\vec{r} = \int_{C\,a}^{b} \overbrace{(3xy\,\hat{i} - y^2\hat{j})}\cdot\overbrace{(dx\,\hat{i} + dy\,\hat{j})}$$

$$= \int_{C\,a}^{b} (3xy\,dx - y^2\,dy)$$

Again along the curve $y = 2x^2$, $dy = 4x\,dx$ and $0 \le x \le 1$. Substituting $\Rightarrow$

II.B.1.) $$\int_{C\,a}^{b} \vec{F}\cdot d\vec{r} = \int_{x=0}^{1} \left[3x(2x^2)dx - (2x^2)^2(4x\,dx)\right]$$

$$= \int_{x=0}^{1} [6x^3 - 16x^5]dx = -7/6$$

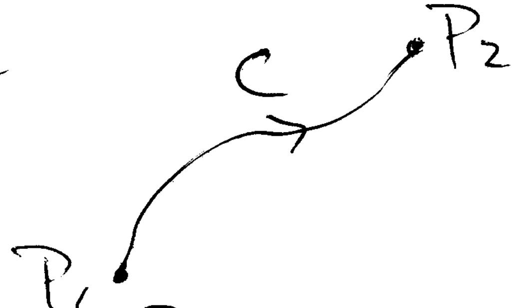
<u>Theorem</u>: If $\vec{F} = \vec{\nabla}\varphi$ everywhere in a region of space R, defined by $a_1 \le x \le a_2$; $b_1 \le y \le b_2$; $c_1 \le z \le c_2$; where $\varphi(x,y,z)$ is single valued and has continuous derivatives in R,

then

$$\int_{C\,a}^{b} \vec{F}\cdot d\vec{l}$$ is independent of the path C in R joining points a and b.

In such a case $\vec{F}$ is called a <u>conservative vector field</u> and φ is its <u>scalar potential</u>.

II.B.1.) **Proof**: Integrate $\vec{F} = \vec{\nabla}\varphi$ from $P_1 = (x_1, y_1, z_1)$ to point $P_2 = (x_2, y_2, z_2)$ in R along C



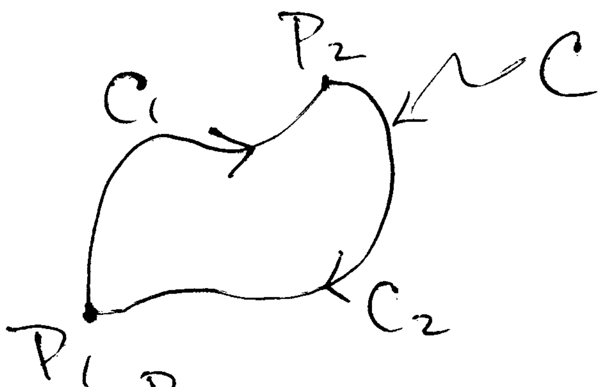
$$\int_{C\,P_1}^{P_2} \vec{F}\cdot d\vec{r} = \int_{C\,P_1}^{P_2} \vec{\nabla}\varphi \cdot d\vec{r} = \int_{\varphi(P_1)}^{\varphi(P_2)} d\varphi$$

$$= \varphi(x_2, y_2, z_2) - \varphi(x_1, y_1, z_1).$$

The value of the integral depends only on the points P_1 and P_2, not the path joining them.

Corollary: $\oint_C \vec{F}\cdot d\vec{l} = 0$ around any closed curve C in R.

II.B.1.)



$C = C_1 + C_2$

But $\int_{P_1 \atop C_1}^{P_2} \vec{F}\cdot d\vec{l} = \varphi(P_2) - \varphi(P_1)$

$\int_{P_2 \atop C_2}^{P_1} \vec{F}\cdot d\vec{l} = \varphi(P_1) - \varphi(P_2)$

adding $\Rightarrow \oint_C \vec{F}\cdot d\vec{l} = 0$.

<u>Converse</u> <u>of</u> <u>Theorem</u>: If $\int_{P_1}^{P_2} \vec{F}\cdot d\vec{l}$ is independent of the path C joining any two points P_1 and P_2, then there exists a function φ such that $\vec{F} = \vec{\nabla}\varphi$.

II B) <u>Converse of Theorem</u>: If $\int_{P_1}^{P_2} \vec{F}\cdot d\vec{l}$ is independent of the path C joining any two points P_1 & P_2 Then there exists a function φ such that

$$\vec{F} = \vec{\nabla}\varphi.$$

<u>Proof</u>: $\vec{F} = F_x\hat{i} + F_y\hat{j} + F_z\hat{k}$

Let $P_1 = (x_1, y_1, z_1)$ & $P_2 = (x, y, z)$

Define

$$\varphi(x,y,z) = \int_{P_1}^{P_2} \vec{F}\cdot d\vec{l}$$

So

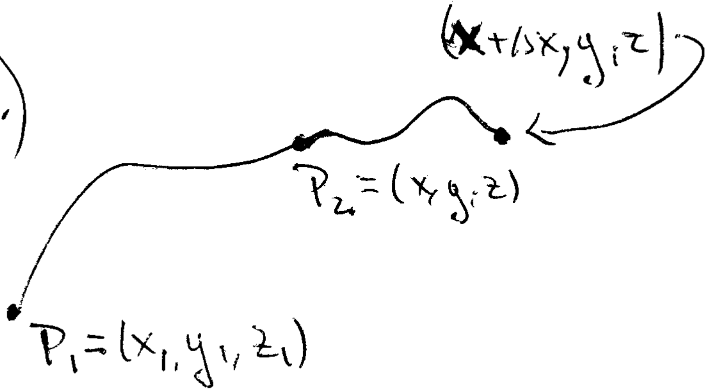
$$\varphi(x+\Delta x, y, z) - \varphi(x, y, z)$$

$$= \int_{P_1}^{(x+\Delta x, y, z)} \vec{F}\cdot d\vec{l} - \int_{P_1}^{P_2} \vec{F}\cdot d\vec{l}$$

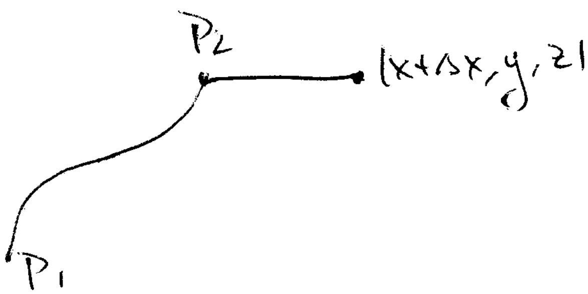
$$= \int_{P_2}^{P_1} \vec{F}\cdot d\vec{l} + \int_{P_1}^{(x+\Delta x, y, z)} \vec{F}\cdot d\vec{l}$$

$$= \int_{P_2}^{(x+\Delta x, y, z)} \vec{F}\cdot d\vec{l}$$

II.B.)



Since $\int$ is indep. of the path choose the path to be a straight line so that $dy = dz = 0$



So
$$\frac{\psi(x+\Delta x, y, z) - \psi(x,y,z)}{\Delta x} = \frac{1}{\Delta x}\int_{P_2=(x,y,z)}^{(x+\Delta x, y, z)} F_x\, dx$$
$$= \frac{1}{\Delta x}\overline{F_x}\,\Delta x = F_x \quad \text{(by mean value thm.)}$$
as $\Delta x \to 0$

$$\boxed{\frac{\partial \psi}{\partial x} = F_x}$$

Similarly $F_y = \frac{\partial \psi}{\partial y}$, $F_z = \frac{\partial \psi}{\partial z}$

So $\boxed{\vec{F} = \vec{\nabla}\psi}$ //.

II.B.)

Theorem: A necessary & sufficient condition for a field $\vec{F}$ to be conservative is that

$$\overrightarrow{\text{Curl}\,\mathbf{F}} = \vec{\nabla}\times\vec{F} = 0.$$

Proof. 1) Necessary: if $\vec{F} = \vec{\nabla}\varphi$

then $\vec{\nabla}\times\vec{F} = \vec{\nabla}\times\vec{\nabla}\varphi = 0.$

2) Sufficiency: If $\vec{\nabla}\times\vec{\mathbf{F}} = 0$

then $\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial_x & \partial_y & \partial_z \\ F_x & F_y & F_z \end{vmatrix} = 0$ So

$$\partial_x \mathbf{F}_z = \partial_z \mathbf{F}_x\ ,\ \partial_z \mathbf{F}_y = \partial_y \mathbf{F}_z\ ,\ \partial_y \mathbf{F}_x = \partial_x \mathbf{F}_y.$$

We must prove $\vec{F} = \vec{\nabla}\varphi$.

So $\displaystyle\int_C \vec{F}\cdot d\vec{l} = \int_C F_x dx + F_y dy + F_z dz$

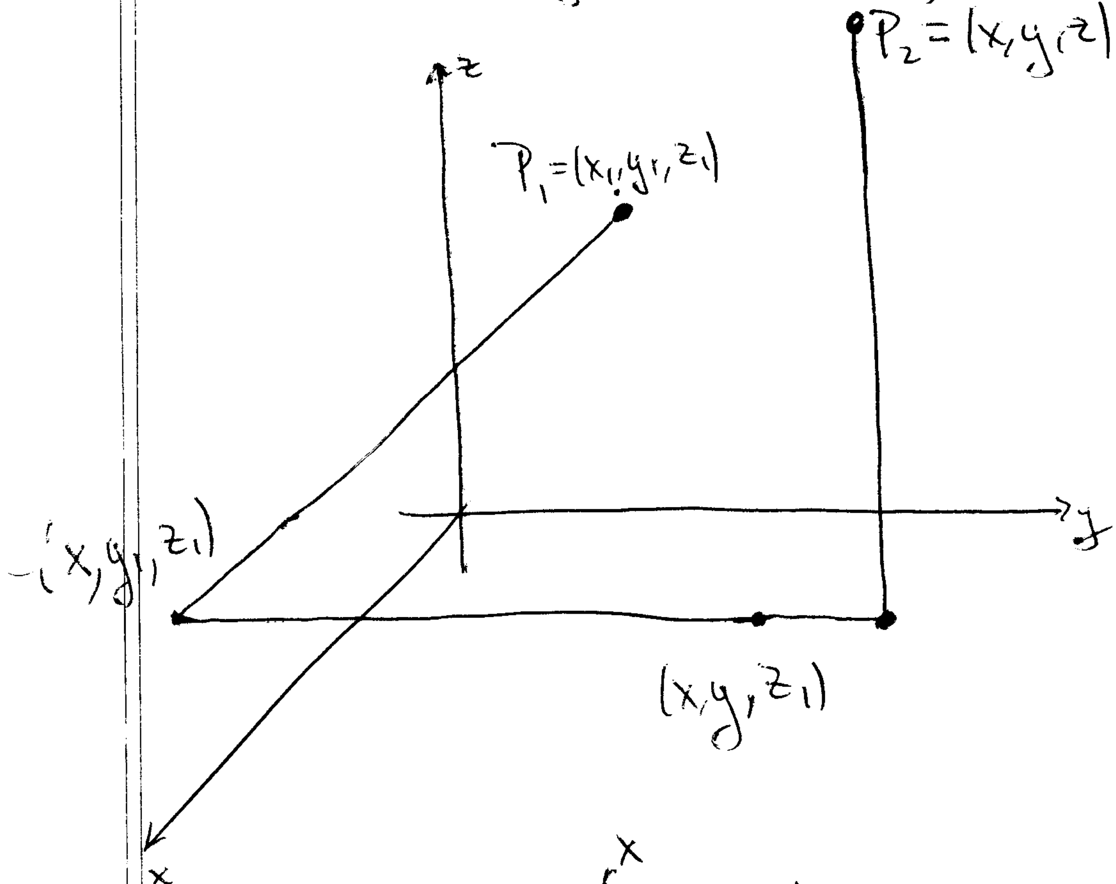
where C is the path joining (x_1, y_1, z_1) to (x, y, z).

Let's choose the path to be straight line segments from

II.B.)

(x_1, y_1, z_1) to (x, y_1, z_1) to (x, y, z_1) to (x, y, z)

and call $\varphi(x,y,z)$ the integral along this path.



So $$\varphi(x,y,z) = \int_{x_1}^{x} F_x(x', y_1, z_1)\,dx'$$

$$+ \int_{y_1}^{y} F_y(x, y', z_1)\,dy'$$

$$+ \int_{z_1}^{z} F_z(x, y, z')\,dz'$$

II.3) Thus

$$\frac{\partial \psi}{\partial z} = F_z(x,y,z)$$

use $\vec{\nabla}\times\vec{F}=0$.

$$\frac{\partial \psi}{\partial y} = F_y(x,y,z_1) + \int_{z_1}^{z} \partial_y F_z(x,y,z')\,dz'$$

$$= F_y(x,y,z_1) + \int_{z_1}^{z} \partial_{z'} F_y(x,y,z')\,dz'$$

$$= F_y(x,y,z_1) + F_y(x,y,z')\Big|_{z'=z_1}^{z}$$

$$= F_y(x,y,z_1) + F_y(x,y,z) - F_y(x,y,z_1)$$

$$= F_y(x,y,z)$$

$$\frac{\partial \psi}{\partial x} = F_x(x,y_1,z_1) + \int_{y_1}^{y} \partial_x F_y(x,y',z_1)\,dy' + \int_{z_1}^{z} \partial_x F_z(x,y,z')\,dz'$$

use $\vec{\nabla}\times\vec{F}=0$

$$= F_x(x,y,z)$$

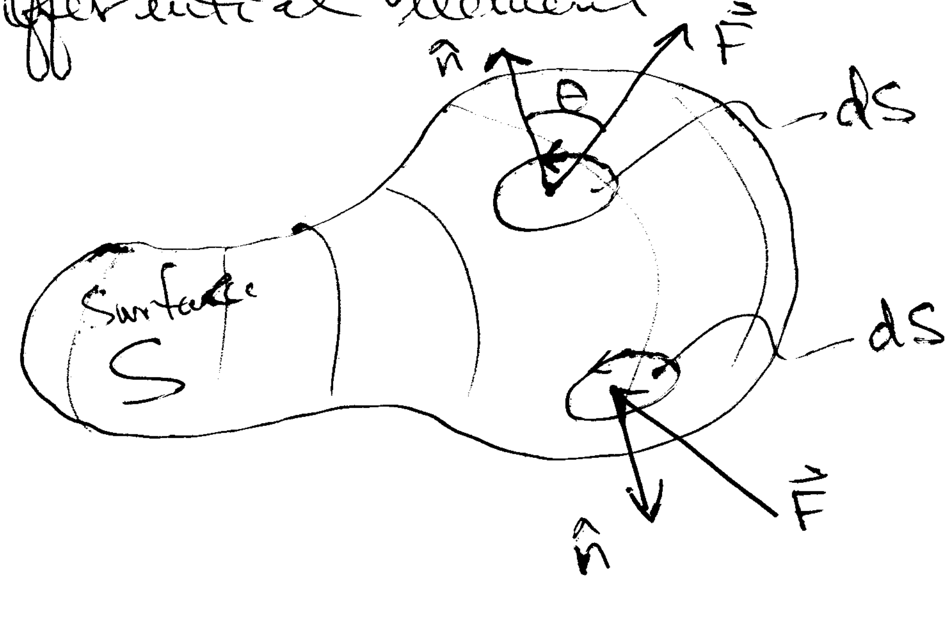
II.B.1.) So

$$\vec{\vec{F}} = \partial_x \varphi \,\hat{i} + \partial_y \varphi \,\hat{j} + \partial_z \varphi \,\hat{k}$$
$$= \vec{\nabla}\varphi \, .//.$$

The other two line integrals are defined analogously.

II.B.2.) Surface Integrals: Flux divergence and solid angles.

Consider integrals over the directed area of a surface next, in particular consider an element of area dS or da upon a surface S. Let $\vec{\vec{F}}$ be the value of a vector field at the center of the differential element



II.B.2.) The component of $\vec{F}$ $\perp$ to the surface at $d\vec{S}$ is the projection of $\vec{F}$ onto the normal $\hat{n}$

$$\vec{F}\cdot\hat{n} = |\vec{F}|\cos\theta$$

Def. The flux of $\vec{F}$ through the element $d\vec{S}$ is $\vec{F}\cdot d\vec{S}$. Let ϕ denote the total flux of $\vec{F}$ through the surface S; ϕ is the surface integral of the normal component of $\vec{F}$ over the surface S:

$$\phi = \iint_S \vec{F}\cdot d\vec{S} \quad .$$

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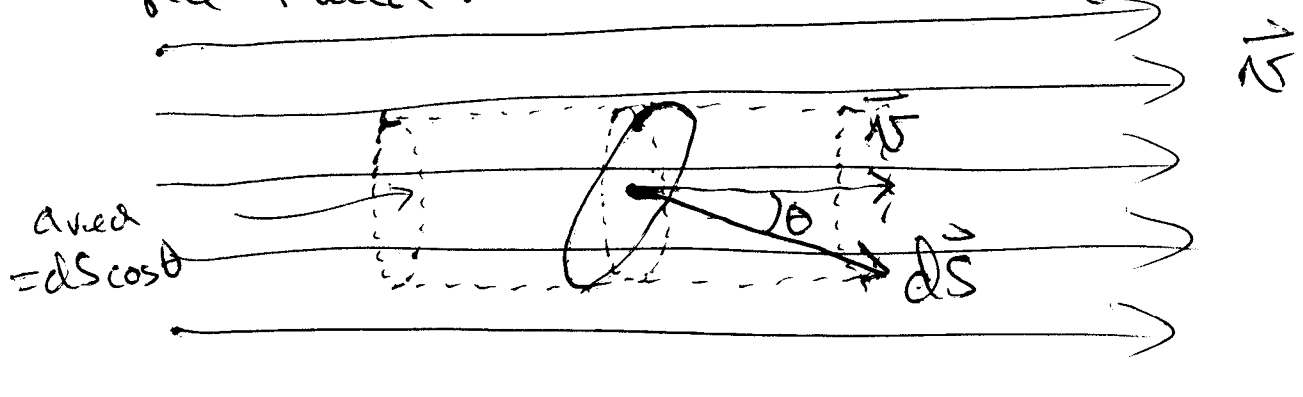
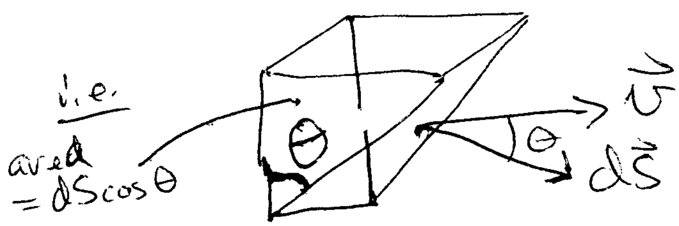
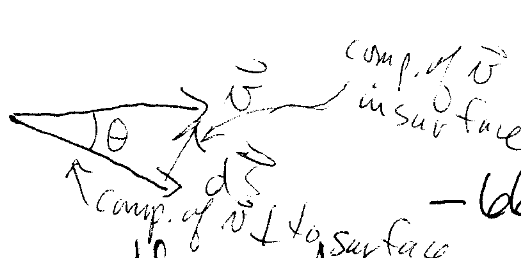
example: To motivate the utility of such an integral in physical problems consider the surface to be in a fluid and $\vec{F} = \vec{v}$, the velocity field of the fluid.







II.B.2.) The volume of fluid flowing through a surface dS per unit time is just

$$|\vec{V}|\cos\theta\, dS = \text{rate of vol. flow}$$
$$= \vec{v}\cdot d\vec{S}$$

(i.e. only the normal component of $\vec{v}$ corresponds to flow through the surface)

The flux is the ~~net~~ total flow ~~in or out of~~ through the surface closed surface

$$\phi = \int_S \vec{v}\cdot d\vec{S}$$

If the fluid density is ρ then the mass flowing through the surface per unit time is $\rho\vec{v}\cdot d\vec{S}$ and the flux is the (~~net~~ total closeds) mass flowing through surface.

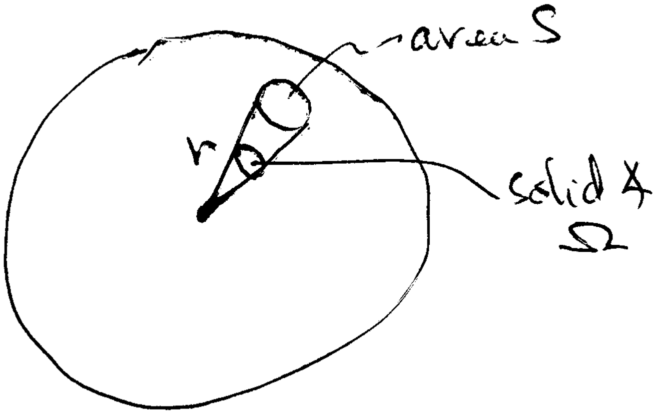
$$\phi = \int_S \rho\vec{v}\cdot d\vec{S}$$

If it is a closed surface, then ϕ is the net loss of or gain of mass in the Rate of volume bounded by the surface.

II.B.2.) Example: An example of flux is the solid angle.

First, for a sphere the solid angle is defined by

$$\Omega \equiv \frac{\text{area}}{(\text{radius})^2} = \frac{S}{r^2}$$



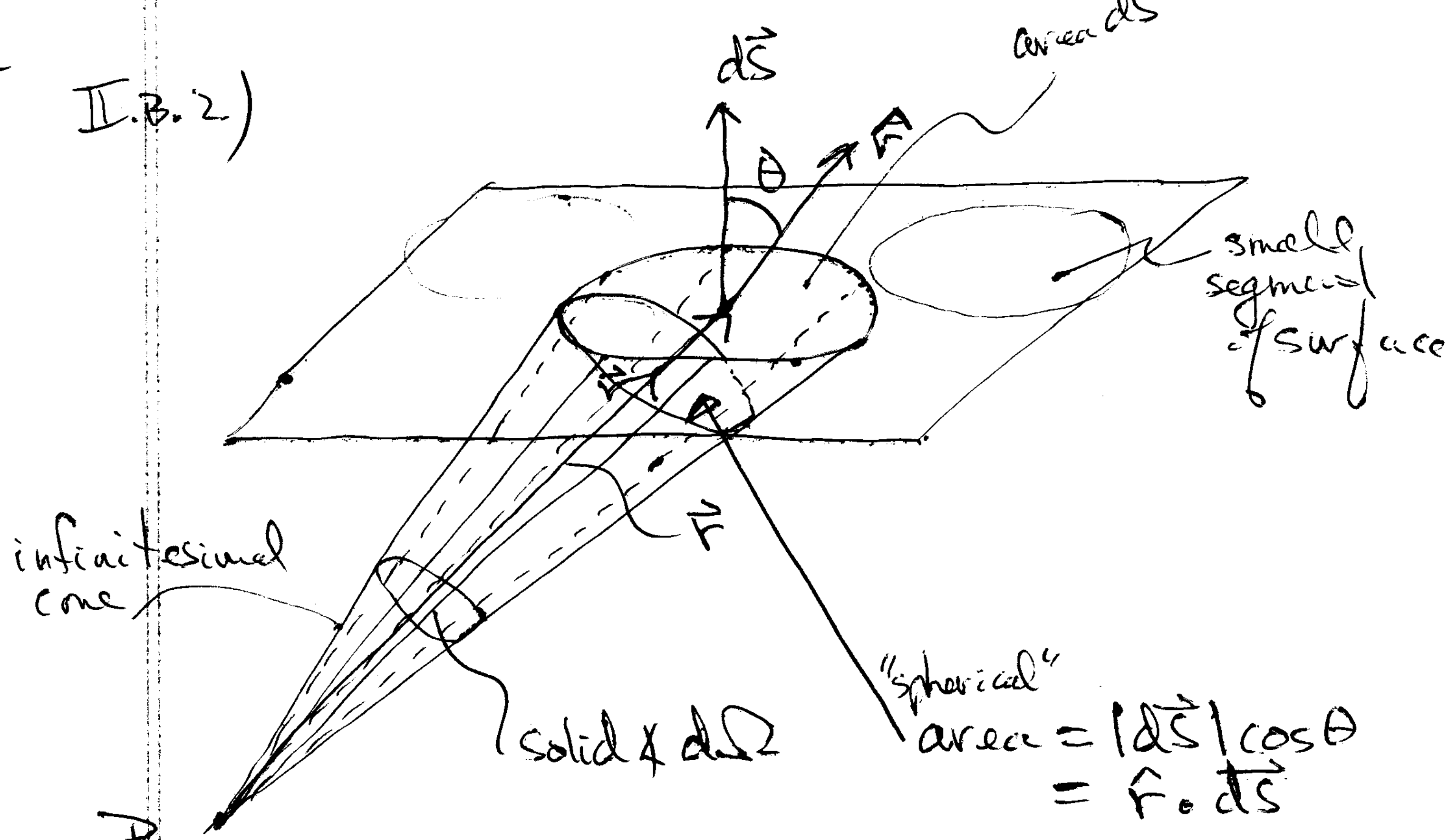
for the full sphere

$$\Omega = \frac{4\pi r^2}{r^2} = 4\pi$$

sterradians.

Next, for an arbitrary surface we can break up the surface into infinitesimal areas, then calculate the solid ∡ for each area (as if it were part of a sphere) from our point in space and then add the ∡'s up.

II.B.2)



So the solid ∡ $d\Omega$ is given as for the sphere

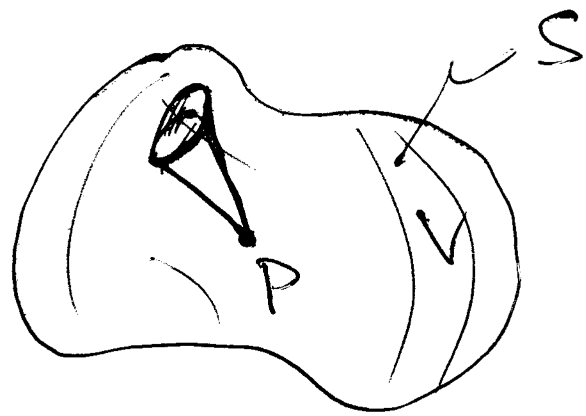
$$d\Omega = \frac{\text{area on sphere}}{r^2} = \frac{|d\vec{S}|\cos\theta}{r^2}$$

$$= \frac{\vec{r}\cdot d\vec{S}}{r^3}$$

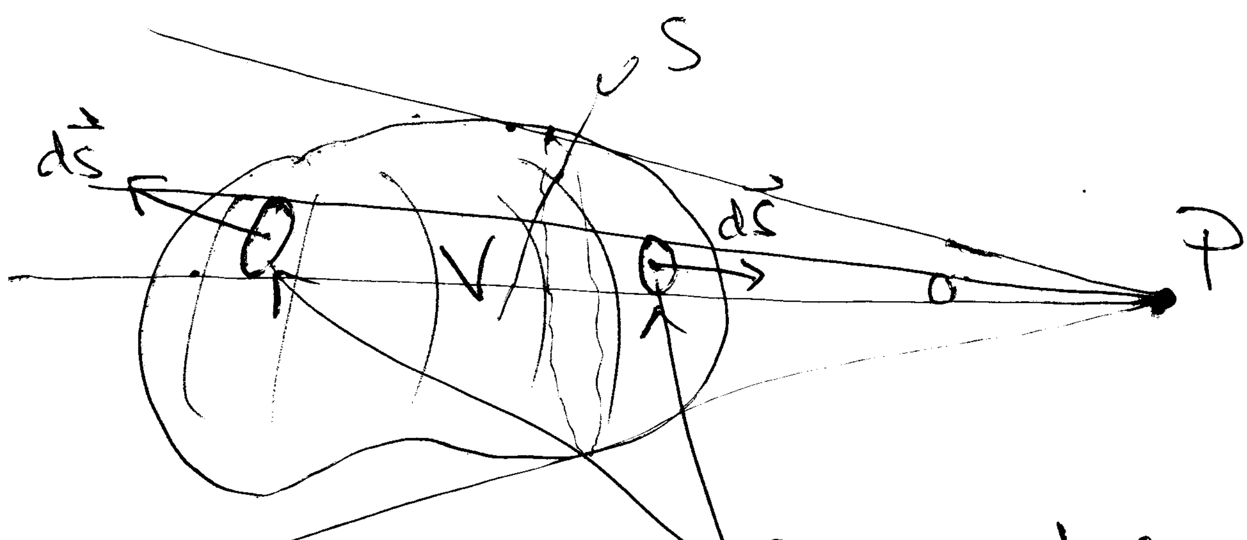
Then for the arbitrary surface S, we add up the infinitesimal solid ∡'s to obtain the solid angle S subtends

$$\Omega = \int_S \frac{\vec{r}\cdot d\vec{S}}{r^3}$$

II.B.2.) For any closed surface and any point P inside the volume V:
$\boxed{\Omega = 4\pi}$ since the whole sphere will be swept out



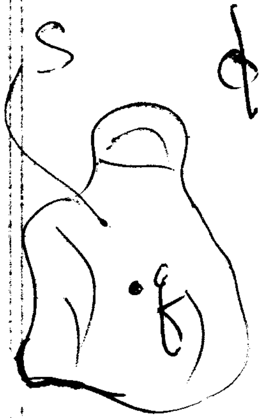
For P outside of V: $\boxed{\Omega = 0}$



Same magnitude
opposite sign
So solid ∡'s cancel.

II B.2.) So the solid $\measuredangle$ = $\oint$ flux of $\frac{\vec{r}}{r^3} = \frac{\hat{r}}{r^2}$.

<u>example</u>: What is the flux of $\vec{E}$ through a surface S enclosing a point charge q?



$$\phi = \int_S \vec{E}\cdot d\vec{S} = \int_S \frac{q}{4\pi\epsilon_0}\frac{\hat{r}}{r^2}\cdot d\vec{S}$$

$$= \frac{q}{4\pi\epsilon_0}\underbrace{\Omega_S}_{=4\pi} = \frac{q}{\epsilon_0}$$

Example of Gauss's Law.

II.B.3.) Next, we will relate the divergence of a vector field and the flux of a vector.

Consider the vector field $\vec{F}(\vec{r})$ (eg. the velocity field of a fluid) and let $\vec{F}(\vec{r})$ be the value of the vector at the center of an infinitesimal element of volume with sides dx, dy, dz and center at $\vec{r}$.