II.) Vector Calculus

A) Differentiation

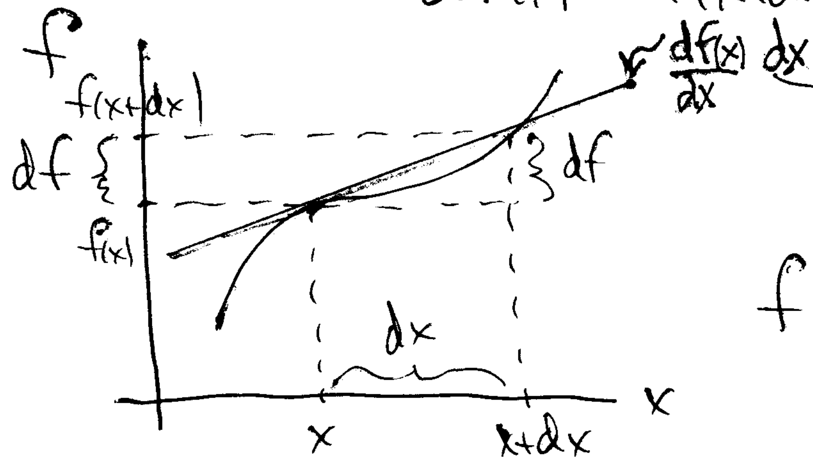
1) Gradient Operator:

Recall the derivative of a scalar function. If $f(x)$ is a function of 1 variable, its derivative, if it exists, is defined as the limit

$$\frac{df(x)}{dx} = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{x+\Delta x - x}.$$

The incremental change in $f(x)$ in going from x to $x+dx$ is just

$$df(x) = f(x+dx) - f(x) = \frac{df(x)}{dx}\, dx$$



This is just Taylor's Theorem

$$f(x+h) = f(x) + h\frac{df}{dx} + \frac{1}{2!}h^2\frac{d^2f}{dx^2} + \cdots$$

where $h = dx$ and we neglect higher order terms.

II.A.1) We can proceed similarly for functions of 3 variables (i) $\varphi = \varphi(x,y,z) = \varphi(\vec{r})$.

Consider the incremental change in φ, $d\varphi$, when going from (x,y,z) to the point $(x+dx, y+dy, z+dz)$. This is just given by Taylor's theorem

$$d\varphi(x,y,z) = \frac{\partial\varphi}{\partial x}dx + \frac{\partial\varphi}{\partial y}dy + \frac{\partial\varphi}{\partial z}dz.$$

We can write this in a convenient form, as the scalar product of 2 vectors:

$$\boxed{d\varphi = \vec{\nabla}\varphi \cdot d\vec{r}}$$

where the arbitrary infinitesimal displacement vector $d\vec{r}$ is

$$d\vec{r} = dx\,\hat{i} + dy\,\hat{j} + dz\,\hat{k}$$

and the <u>gradient</u> <u>vector</u> is given by

$$\vec{\nabla}\varphi \equiv \frac{\partial\varphi}{\partial x}\hat{i} + \frac{\partial\varphi}{\partial y}\hat{j} + \frac{\partial\varphi}{\partial z}\hat{k}.$$

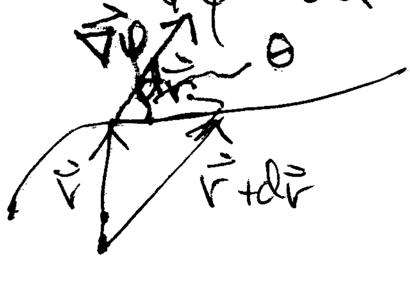
II.A.1) Since φ is a scalar function so is $d\varphi$, and $d\vec{r}$ is a vector; then indeed $\vec{\nabla}\varphi$ is a vector function. The differential operator

$$\vec{\nabla} = \hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}$$

is called a vector differential operator. It has the special names "gradient" operator, "del" operator, nabla (= Syrian harp in the shape ∇).

$\vec{\nabla}\varphi$ is the "gradient" of a scalar function φ. Since $\vec{\nabla}\varphi$ is a vector function, it has a geometrical meaning independent of the Cartesian coordinate system we have used to motivate its definition. Towards this coordinate independent characterization of the gradient, consider the directional derivative. Since

$$d\varphi = \vec{\nabla}\varphi \cdot d\vec{r} = |\vec{\nabla}\varphi||d\vec{r}|\cos\theta$$



II.A.1) So the <u>directional derivative</u> of φ in the direction of $d\vec{r}$ is defined as

$$\frac{d\varphi}{|d\vec{r}|} \equiv |\vec{\nabla}\varphi| \cos\theta \quad \text{with } \theta = \measuredangle \text{ between } \vec{\nabla}\varphi \;\&\; d\vec{r}.$$

Thus for a fixed small distance $|d\vec{r}|$ from the point $\vec{r}$, the change in φ is maximum when $d\vec{r}$ is in the same direction as $\vec{\nabla}\varphi$.

(*) geo. def. of $\vec{\nabla}\varphi$: Turning this around, we can define $\vec{\nabla}\varphi$ as the vector having at any point the magnitude equal to the most rapid rate of increase of φ at the point and being in the direction of this fastest rate of increase.

Hence we can summarize this by

1) $\vec{\nabla}\varphi$ is defined by (*)

2) $d\varphi = \vec{\nabla}\varphi \cdot d\vec{r}$

3) In a Cartesian Coordinate system

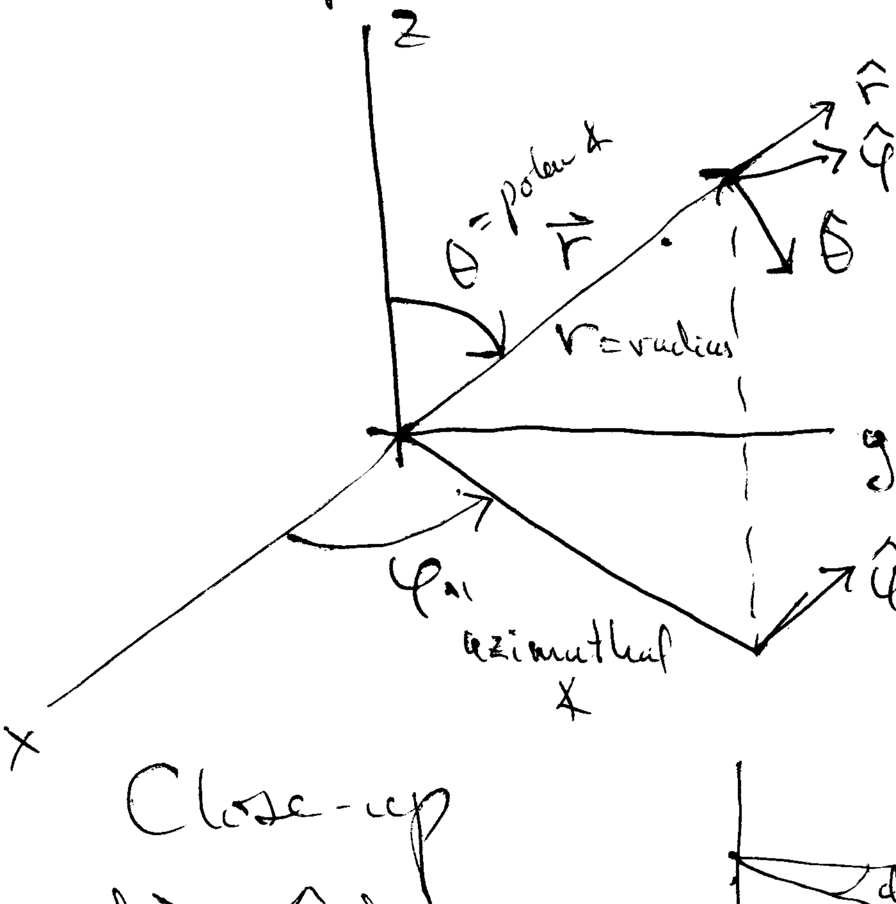
II.A.1)

3) $$\vec{\nabla}\varphi = \frac{\partial\varphi}{\partial x}\hat{i} + \frac{\partial\varphi}{\partial y}\hat{j} + \frac{\partial\varphi}{\partial z}\hat{k}$$

$$\left(= \partial_x\varphi\,\hat{i} + \partial_y\varphi\,\hat{j} + \partial_z\varphi\,\hat{k} \right)$$

4) In other coordinate systems, we find $d\varphi$ and $d\vec{r} \Rightarrow \vec{\nabla}\varphi$

ex. Spherical Polar Coordinates

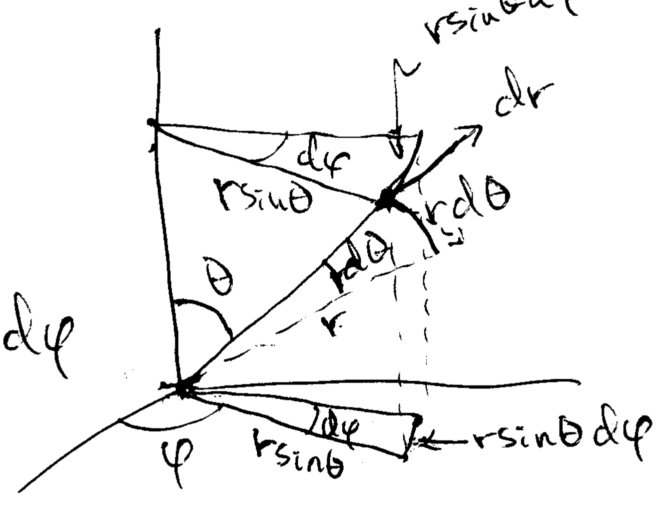


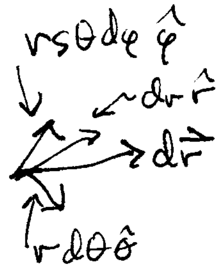
$$\left.\begin{aligned} \hat{a}_r &= \hat{u}_1 = \hat{r} \\ \hat{a}_\theta &= \hat{u}_2 = \hat{\theta} \\ \hat{a}_\varphi &= \hat{u}_3 = \hat{\varphi} \end{aligned}\right\} \text{mutually } \perp \text{ unit basis vectors}$$

$$x = r\sin\theta\cos\varphi$$
$$y = r\sin\theta\sin\varphi$$
$$z = r\cos\theta$$

Close-up

$$d\vec{r} = \hat{r}\,dr + \hat{\theta}\,r\,d\theta + \hat{\varphi}\,r\sin\theta\,d\varphi$$





II.A.1)(4.) Now (use function ϕ here, not to confuse it with $\sphericalangle\varphi$)

$$d\phi = \vec{\nabla}\phi \cdot d\vec{r}$$

$$\equiv \left[(\vec{\nabla}\phi)_r \hat{r} + (\vec{\nabla}\phi)_\theta \hat{\theta} + (\vec{\nabla}\phi)_\varphi \hat{\varphi} \right] \cdot$$
$$\cdot \left[\hat{r}\,dr + \hat{\theta}\,r\,d\theta + \hat{\varphi}\,r\sin\theta\,d\varphi \right]$$

$$= (\vec{\nabla}\phi)_r dr + (\vec{\nabla}\phi)_\theta r\,d\theta + (\vec{\nabla}\phi)_\varphi r\sin\theta\,d\varphi$$

On the other hand $\phi = \phi(r,\theta,\varphi)$, so use Taylor's Theorem

$$d\phi = \frac{\partial\phi}{\partial r} dr + \frac{\partial\phi}{\partial\theta} d\theta + \frac{\partial\phi}{\partial\varphi} d\varphi$$

but $dr, d\theta, d\varphi$ are indep. $\Rightarrow$

$$(\vec{\nabla}\phi)_r = \frac{\partial\phi}{\partial r} \quad ; \quad (\vec{\nabla}\phi)_\theta = \frac{1}{r}\frac{\partial\phi}{\partial\theta}$$

$$(\vec{\nabla}\phi)_\varphi = \frac{1}{r\sin\theta}\frac{\partial\phi}{\partial\varphi}$$

Hence

II A.1.4.) in spherical polar Coordinates

$$\boxed{\vec{\nabla}\phi = \frac{\partial \phi}{\partial r}\hat{r} + \frac{1}{r}\frac{\partial \phi}{\partial \theta}\hat{\theta} + \frac{1}{r\sin\theta}\frac{\partial \phi}{\partial \varphi}\hat{\varphi}}.$$

II. A.2.) Besides using the vector differential operator $\vec{\nabla}$ to make a vector function from a scalar function ϕ; i.e. $\vec{\nabla}\phi$, we can let $\vec{\nabla}$ act on vector functions using the scalar product to make a scalar function, or the cross product to make a vector function.

Let $\vec{F}(\vec{r})$ be a vector function of the position vector $\vec{r}$!

a.) Divergence: the scalar function called the divergence of $\vec{F}$ can be obtained by taking the dot product of $\vec{\nabla}$ with $\vec{F}$:

$$\vec{\nabla}\cdot\vec{F} = \left[\hat{i}\,\partial_x + \hat{j}\,\partial_y + \hat{k}\,\partial_z\right]\cdot \left[F_x\hat{i} + F_y\hat{j} + F_z\hat{k}\right]$$

II A.2 a.) $$\boxed{\vec{\nabla}\circ\vec{F} = \partial_x F_x + \partial_y F_y + \partial_z F_z}$$

b.) <u>Curl</u>: The <u>curl of $\vec{F}$</u> is the vector function made from the cross product of $\vec{\nabla}$ with $\vec{F}$.

$$\vec{\nabla}\times\vec{F} \equiv \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial_x & \partial_y & \partial_z \\ F_x & F_y & F_z \end{vmatrix}$$

$$= \hat{i}\left(\partial_y F_z - \partial_z F_y\right) + \hat{j}\left(\partial_z F_x - \partial_x F_z\right) + \hat{k}\left(\partial_x F_y - \partial_y F_x\right).$$

As with the gradient, these operations have geometrical definitions, and hence have integrity irrespective of the coordinate system used to calculate them. We will explore these definitions with the aid of <u>Vector Integration</u>.