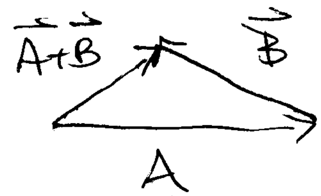
I.B.6) vi) <u>Triangle Inequality</u>

$$|\vec{A}+\vec{B}| \leq |\vec{A}| + |\vec{B}|$$

This follows from the sum of any 2 sides of a triangle is greater than or equal to the third side



I.B.6) vii) $\vec{O}$ is the additive identity

$$\vec{A}+\vec{O} = \vec{A}$$

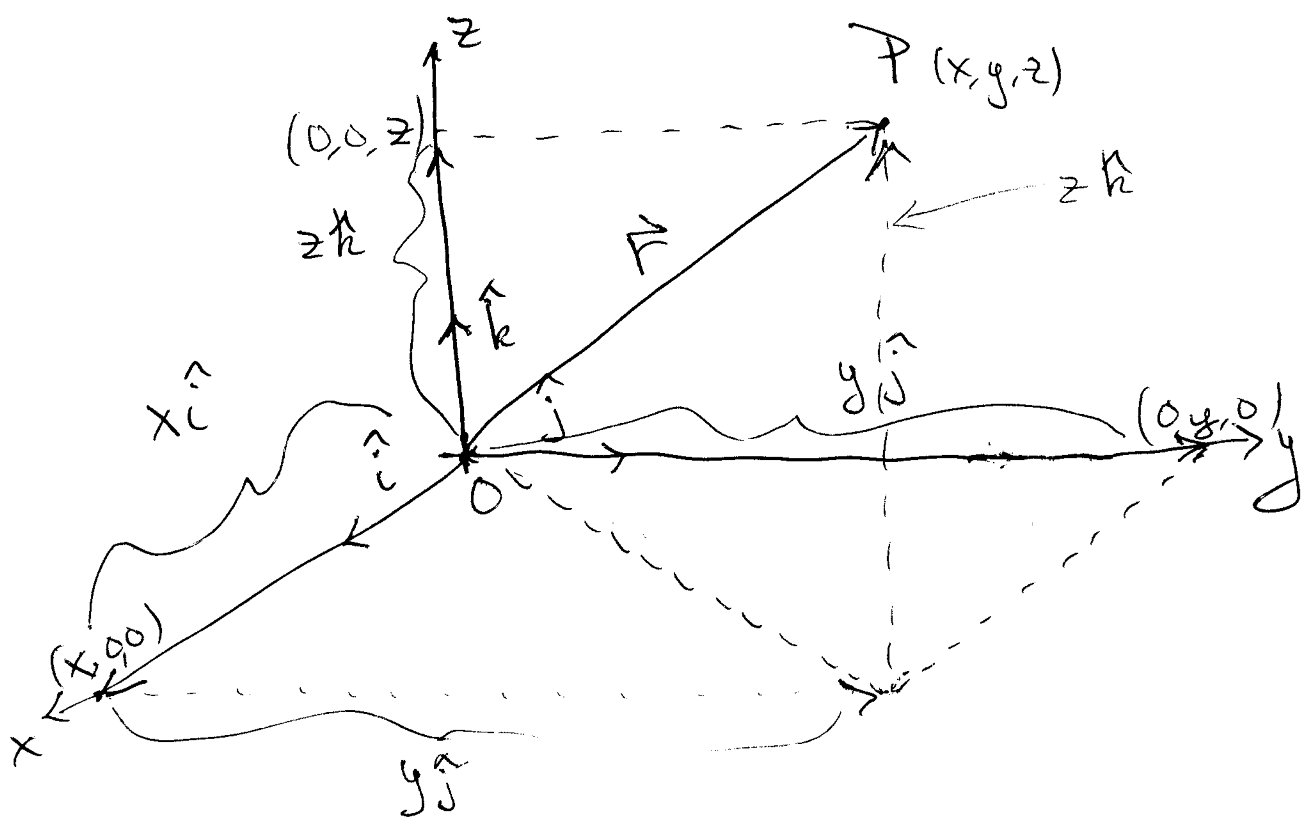
since $\vec{O}$ has tail and head the same.

I.) C.) <u>Cartesian Coordinate System</u> (components)

Definition: The vectors $\vec{V}_1, \vec{V}_2, \ldots, \vec{V}_N$ are <u>linearly dependent</u> if we can find numbers $\alpha_1, \ldots, \alpha_N$, <u>not all zero</u>, such that $\alpha_1\vec{V}_1 + \alpha_2\vec{V}_2 + \cdots + \alpha_N\vec{V}_N = 0$.

I.C.) If this can only be satisfied for $\alpha_1 = \alpha_2 = \cdots = \alpha_N = 0$, then the set of vectors $\vec{V}_1, \ldots, \vec{V}_N$ is a <u>linearly independent</u> set.

In 3 dimensions any set of 4 vectors is linearly dependent. Hence choose the linearly independent set of 3 (orthonormal) vectors $\hat{i}, \hat{j}, \hat{k}$ of unit magnitude and along the positive x, y, z axes respectively (+(x, y, z) axes obey right hand rule as do ($\hat{i}, \hat{j}, \hat{k}$)). This forms a Cartesian Coordinate System. We can express any vector as a linear combination of the 3 $\hat{i}, \hat{j}, \hat{k}$ vectors



I C.) Thus vector $\vec{OP} = x\hat{i} + y\hat{j} + z\hat{k}$
it has magnitude

$$|\vec{OP}| = \sqrt{x^2+y^2+z^2} \equiv r^2 = |\vec{r}|^2$$

notation: $\vec{r} \equiv \vec{OP} = x\hat{i} + y\hat{j} + z\hat{k}$.

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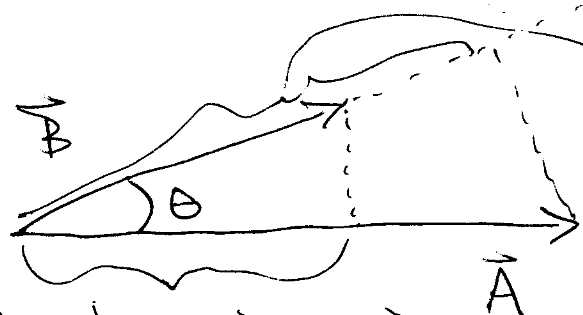
I.D.) <u>Products of Vectors</u>

1) Scalar or dot product of 2 vectors

Let $\vec{A}$ and $\vec{B}$ be 2 vectors and let θ be the angle between them. Then the <u>scalar</u> product of $\vec{A}$ & $\vec{B}$, denoted by $\vec{A}\cdot\vec{B}$, is

$$\vec{A}\cdot\vec{B} = |\vec{A}||\vec{B}|\cos\theta$$



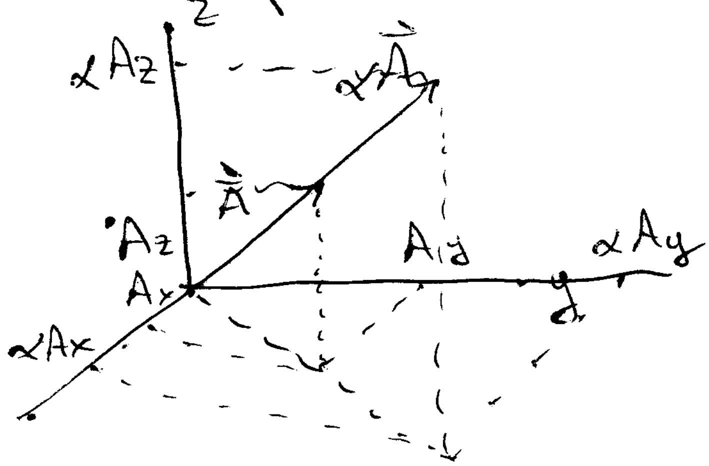
projection of $\vec{A}$ along $\vec{B}$ $= \dfrac{\vec{A}\cdot\vec{B}}{|\vec{B}|} = |\vec{A}|\cos\theta$

Projection of $\vec{B}$ along $\vec{A}$ $= \dfrac{\vec{A}\cdot\vec{B}}{|\vec{A}|} = |\vec{B}|\cos\theta$

a) If $\vec{A} \perp \vec{B} \iff \vec{A}\cdot\vec{B} = 0$

b) Commutative Law $\vec{A}\cdot\vec{B} = \vec{B}\cdot\vec{A}$

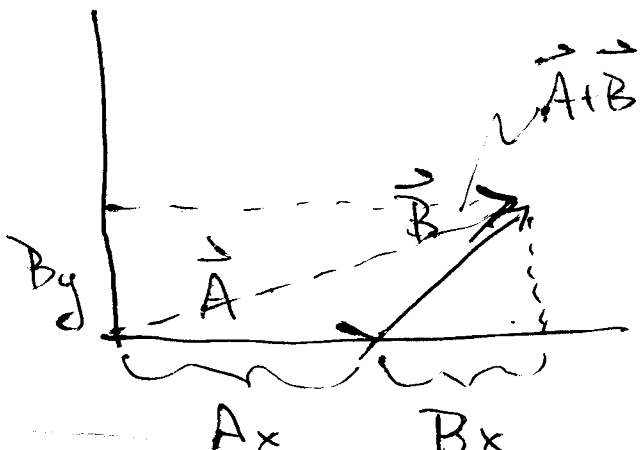
The basic operations of scalar multiplication and vector addition have a simple form in components.



So if $\vec{A} = A_x\hat{i} + A_y\hat{j} + A_z\hat{k}$

$$\boxed{\alpha\vec{A} = \alpha A_x\hat{i} + \alpha A_y\hat{j} + \alpha A_z\hat{k}}$$

Likewise, wlog

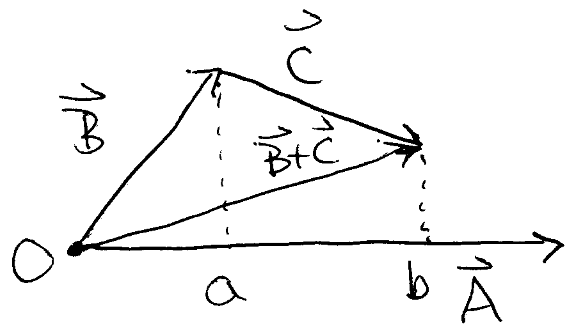


$$\boxed{\vec{A}+\vec{B} = (A_x+B_x)\hat{i} + (A_y+B_y)\hat{j} + (A_z+B_z)\hat{k}}$$

I.D.1c) Distributive Law for the Scalar Product

$$\vec{A}\cdot(\vec{B}+\vec{C}) = \vec{A}\cdot\vec{B} + \vec{A}\cdot\vec{C}$$

proof:



$$\vec{A}\cdot(\vec{B}+\vec{C}) = |\vec{A}|\,\overline{Ob}$$

$$= |\vec{A}|\left(\overline{Oa} + \overline{ab}\right)$$

$$= \vec{A}\cdot\vec{B} + \vec{A}\cdot\vec{C} \quad //.$$

(since $\vec{A}\cdot\vec{B} = |\vec{A}|\,\overline{Oa}$ and $\vec{A}\cdot\vec{C} = |\vec{A}|\,\overline{ab}$)

d) Scalar Multiplication: α = scalar

$$\alpha(\vec{A}\cdot\vec{B}) = (\alpha\vec{A})\cdot\vec{B} = \vec{A}\cdot(\alpha\vec{B}) = (\vec{A}\cdot\vec{B})\alpha$$

e) Cartesian Coordinate System

$$\hat{i}\cdot\hat{i} = \hat{j}\cdot\hat{j} = \hat{k}\cdot\hat{k} = 1$$

$$\hat{i}\cdot\hat{j} = \hat{i}\cdot\hat{k} = \hat{j}\cdot\hat{k} = 0$$

I. D) 1. e) $\{\hat{i}, \hat{j}, \hat{k}\}$ are an orthonormal set of basis vectors.

Hence expanding any vectors in terms of them

$$\vec{A} = A_x \hat{i} + A_y \hat{j} + A_z \hat{k}$$
$$\vec{B} = B_x \hat{i} + B_y \hat{j} + B_z \hat{k}$$

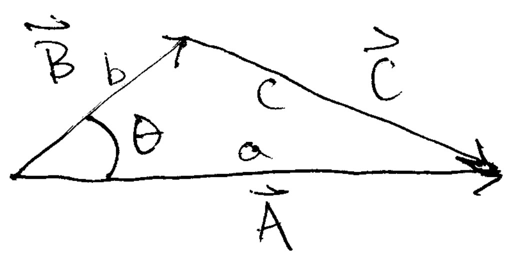
we find

$$\boxed{\vec{A} \cdot \vec{B} = A_x B_x + A_y B_y + A_z B_z}$$

$\hat{i} \cdot \vec{A} = A_x$ projection of $\vec{A}$ on x-axis
$\hat{j} \cdot \vec{A} = A_y$ " " " " y-axis
$\hat{k} \cdot \vec{A} = A_z$ " " " " z-axis

f) Example: Cosine law for plane triangles

$$\vec{A} = \vec{B} + \vec{C} \Rightarrow \vec{C} = \vec{A} - \vec{B}$$



$$\vec{C} \cdot \vec{C} = (\vec{A} - \vec{B}) \cdot (\vec{A} - \vec{B})$$
$$= \vec{A} \cdot \vec{A} + \vec{B} \cdot \vec{B} - 2 \vec{A} \cdot \vec{B}$$

$$\Rightarrow \quad |\vec{C}|^2 = |\vec{A}|^2 + |\vec{B}|^2 - 2|\vec{A}||\vec{B}| \cos\theta$$

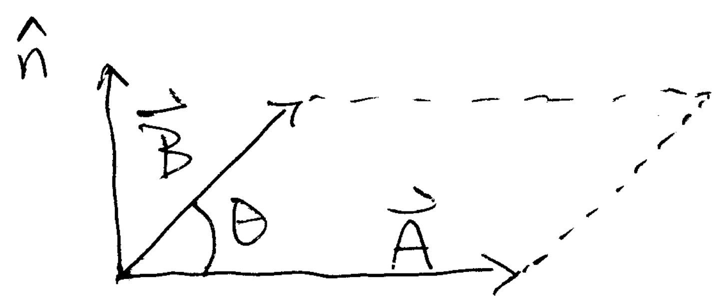
i.e. $$\boxed{c^2 = a^2 + b^2 - 2ab\cos\theta}$$

I.D.2.) Vector Product (Cross product or Wedge product)

Consider 2 vectors $\vec{A}$ and $\vec{B}$, then the cross product is defined as

$$\vec{A}\times\vec{B} \left(= \vec{A}\wedge\vec{B}\right) = |\vec{A}|\,|\vec{B}|\sin\theta\ \hat{n}$$

where $0 \le \theta \le \pi$ and



$\hat{n}$ is the unit vector normal to the plane defined by $\vec{A}$ and $\vec{B}$ and directed such that the right hand rule holds (i.e. $\vec{A}$ rotated into $\vec{B}$ thru $\measuredangle\,\theta$ with fingers of right hand gives thumb pointing in the direction of $\hat{n}$)

Properties of the vector product

a) $\vec{A}\times\vec{B} = -\vec{B}\times\vec{A}$ The cross product is <u>not</u> commutative.

I.D.2 b) Distributive law

$$\vec{A} \times (\vec{B} + \vec{C}) = \vec{A} \times \vec{B} + \vec{A} \times \vec{C}$$

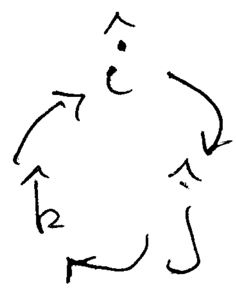
c) Scalar Multiplication : α = scalar

$$\alpha(\vec{A} \times \vec{B}) = (\alpha\vec{A}) \times \vec{B} = \vec{A} \times (\alpha\vec{B}) = (\vec{A} \times \vec{B})\alpha$$

d) Cartesian Coordinate System

$$\hat{i} \times \hat{i} = \hat{j} \times \hat{j} = \hat{k} \times \hat{k} = 0$$

$$\hat{i} \times \hat{j} = \hat{k}$$
$$\hat{j} \times \hat{k} = \hat{i}$$
$$\hat{k} \times \hat{i} = \hat{j}$$



obtain these products by cyclic permutations of one.

Expanding vectors in terms of their components

$$\vec{A} = A_x \hat{i} + A_y \hat{j} + A_z \hat{k}$$
$$\vec{B} = B_x \hat{i} + B_y \hat{j} + B_z \hat{k}$$

we find

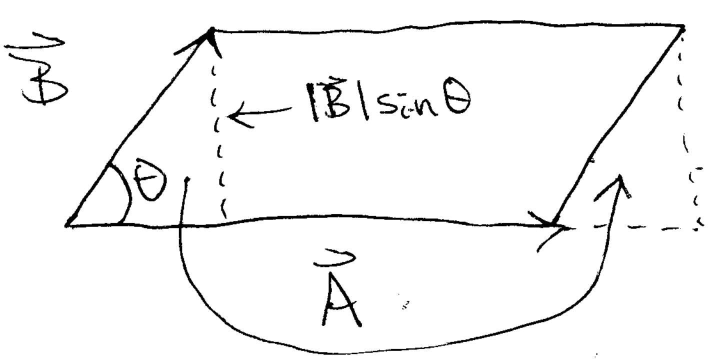
I.D.2d) $$\vec{A}\times\vec{B} = (A_yB_z - A_zB_y)\,\hat{i} + (A_zB_x - A_xB_z)\,\hat{j} + (A_xB_y - A_yB_x)\,\hat{k}$$

which can be conveniently written as

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix}$$

I.D.2 e) Directed Areas (Vector Areas or Projected Areas)

1) Note that $|\vec{A}||\vec{B}|\sin\theta$ is the area of the parallelogram constructed from $\vec{A}$ & $\vec{B}$



Area $= |\vec{A}||\vec{B}|\sin\theta$

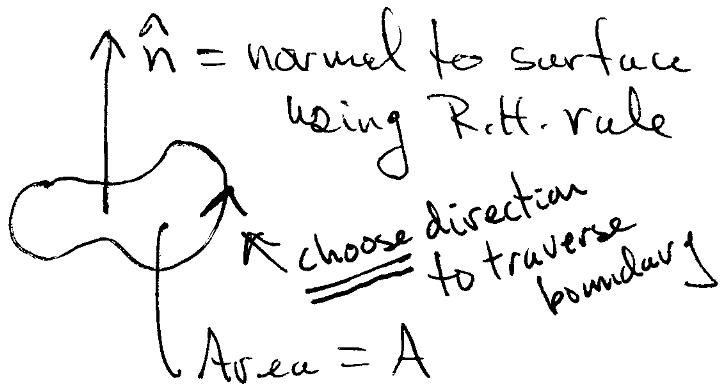
I.D.2e1) Thus we can introduce a <u>directed area</u> for the ▱ :

$$\vec{S} \equiv \vec{A} \times \vec{B}$$

$$\text{area of } ▱ = |\vec{S}| = |\vec{A}||\vec{B}| \sin\theta$$

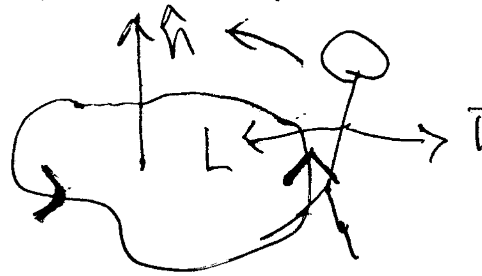
(orientation) direction of ▱ = $\hat{n}$ = right hand rule normal to $\vec{A}, \vec{B}$ (▱) plane

2) For any (orientable) plane surface we can assign a directed area $\vec{S}$



$$\vec{S} \equiv A\,\hat{n}$$

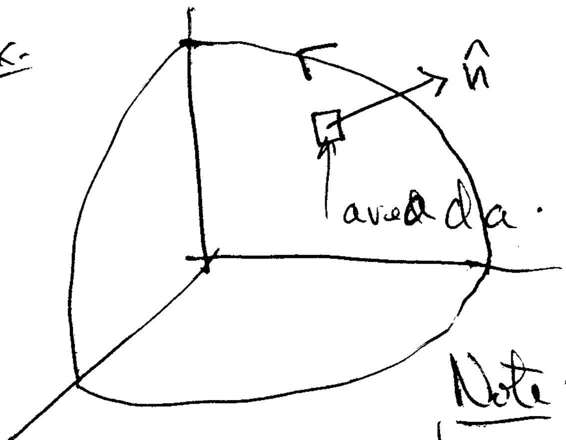
(Right hand Rule for orientable surface



1) choose boundary direction
2) Right hand screw $\equiv \mathcal{E}$)

I.D.2e3) Infinitesimal surface area can be used for directed area

ex.



$d\vec{S} = \hat{n}\, da$

da = infinitesimal element of surface area

Note: for closed surfaces choose $\hat{n}$ to point out.

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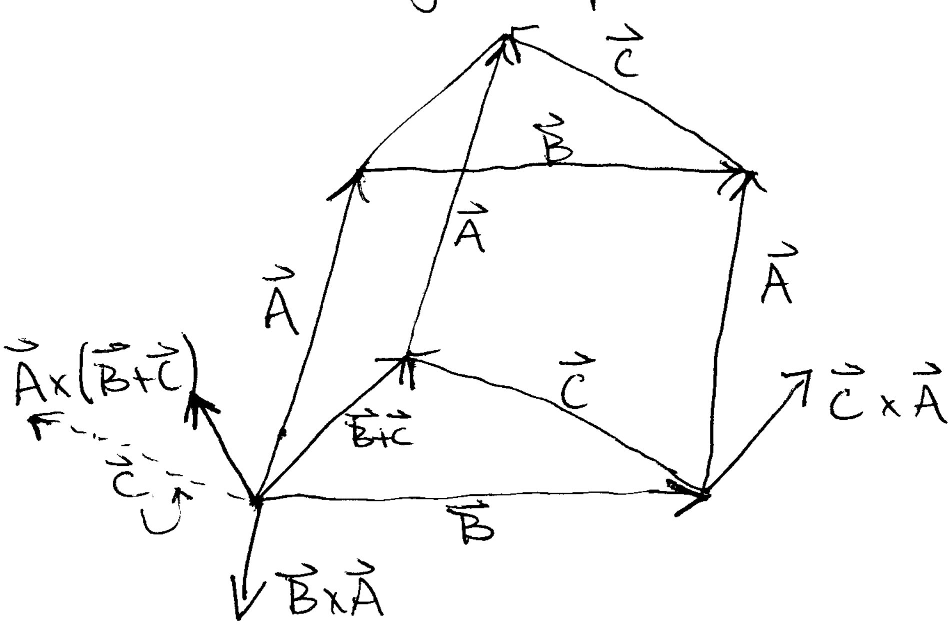
I.D. f) Examples:

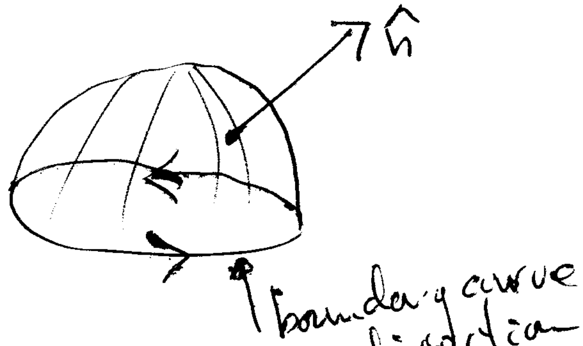
(do f.(2) first)

1) (Proof of Distributive Law) The sum of vector areas of all faces of any polyhedron is zero

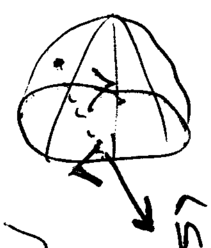
Consider triangular prism



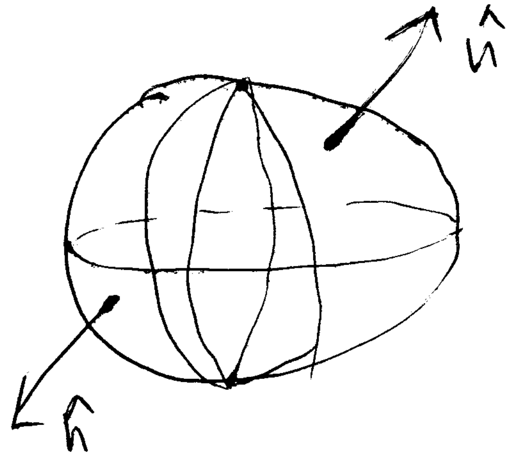
The direction of the normal to a surface is a matter of convention. For open surfaces a direction about which its boundary is traversed is first chosen - this is arbitrary. Then we choose the normal direction to be that of the progress of a right handed screw as turned in the direction of the boundary curve ($\equiv$ L R). These are called positive orientation



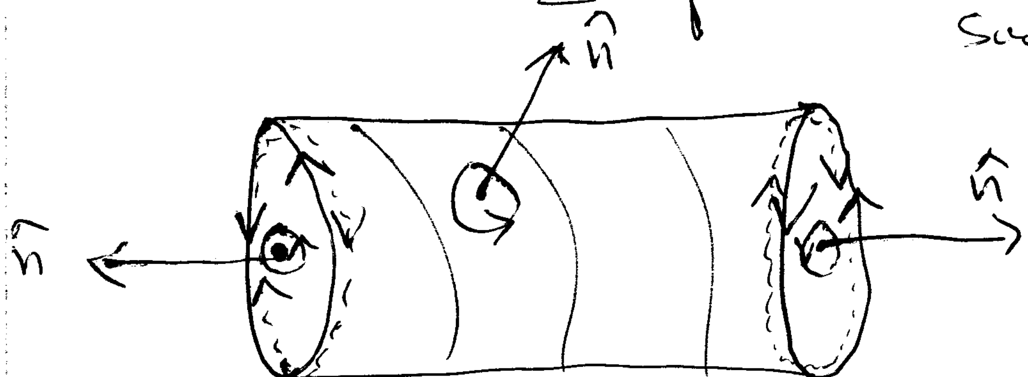
if we chose opposite boundary curve direction then $\hat{n}$ is



The convention for closed surfaces is that the normal points outward



for surfaces where normal field is not smooth ex. piecewise smooth surfaces



The normals still point out, but imaginary boundary curves go in opposite directions consistent with RH screw rule.

These are conventions that will lead to consistent integral theorems.

I.D.2.f.1.)

$$\sum \Delta \text{ end faces area} = \tfrac{1}{2}\vec{B}\times\vec{C} \quad \text{(upper)}$$
(outward facing normals, RH rule)
$$+\ \tfrac{1}{2}\vec{C}\times\vec{B} \quad \text{(lower)}$$
$$\text{Sum} = 0$$

$$\sum \square \text{ areas} = \vec{B}\times\vec{A} + \vec{C}\times\vec{A} + \vec{A}\times(\vec{B}+\vec{C})$$
(outward facing normals, RH rule)

Since $\sum \overrightarrow{\text{area}} = 0 \Rightarrow$

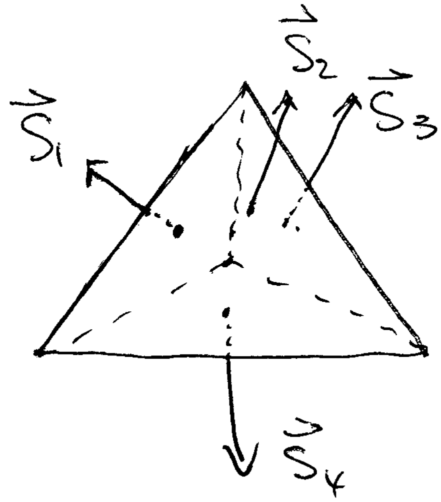
$$\vec{A}\times(\vec{B}+\vec{C}) = -\vec{B}\times\vec{A} - \vec{C}\times\vec{A}$$
$$= \vec{A}\times\vec{B} + \vec{A}\times\vec{C}$$

OR since distribative law is valid

$$\sum \overrightarrow{\text{area}} = 0 .$$

I.D.2.f.2) Consider the tetrahedron

I.D.2.f.2.)



$$\sum_i \overrightarrow{\text{areas}} = 0 = \vec{S}_1 + \vec{S}_2 + \vec{S}_3 + \vec{S}_4$$

Physical Proof: Suppose tetrahedron is an imaginary surface in a fluid in equilibrium under hydrostatic pressure (i.e. pressure inside = pressure outside tetrahedron). Each face experiences a force normal to its plane and proportional to the area $\propto \vec{S}_i$.

But the fluid inside is in equilibrium with the fluid outside, so the sum of these forces must be zero. Since the pressure is the same on all faces, the sum of the vector area = 0.//

Now any polyhedral figure can be divided up into tetrahedra. Internal faces appear twice and areas cancel.

I. D. 2.f.2.) So the total vector area for any polyhedral surface $= 0$. Making the faces smaller and smaller we find for any closed curved surface

$$\iint_S d\vec{S} = 0 .$$

I. D. 3.) <u>Vector Triple Products</u>

a) Vector Triple Product

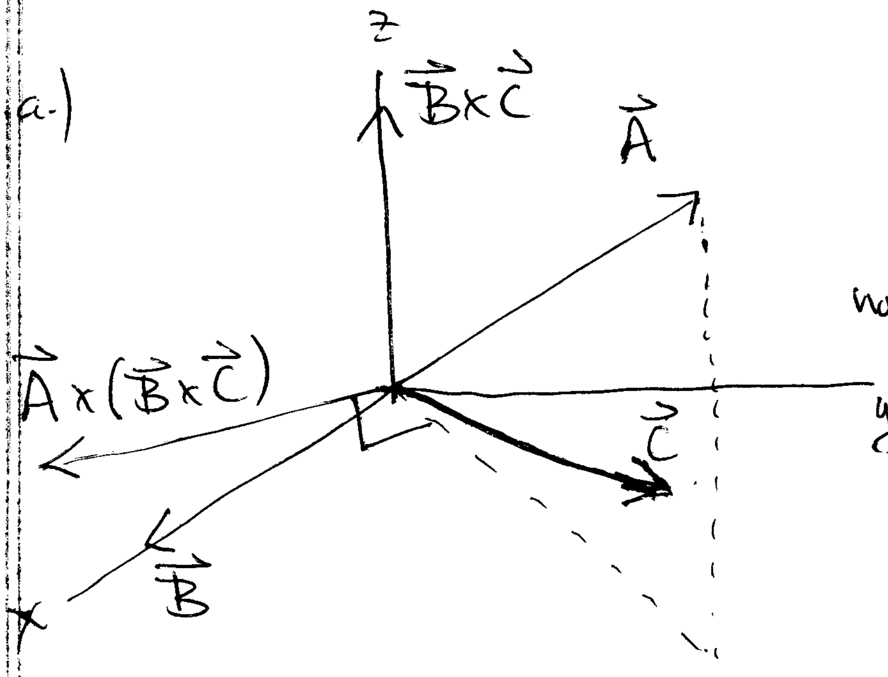
$$\vec{A} \times (\vec{B} \times \vec{C}) = \vec{B}(\vec{A} \cdot \vec{C}) - \vec{C}(\vec{A} \cdot \vec{B})$$

<u>Proof</u>: LHS is $\perp$ to $\vec{A}$ and $(\vec{B} \times \vec{C})$

$\therefore$ $\vec{A} \times (\vec{B} \times \vec{C})$ lies in the plane of $\vec{B}$ and $\vec{C}$

Thus the vectors $\vec{A} \times (\vec{B} \times \vec{C})$, $\vec{B}$ and $\vec{C}$ are <u>linearly dependent</u> since they lie in the same plane.

I.D.3(a.)



wlog choose $\vec{B}$ along x-axis; and note $\vec{B}$, $\vec{C}$ ~~lie~~ in same plane, choose as x-y plane

So
$$\vec{A}\times(\vec{B}\times\vec{C}) = \beta\vec{B} + \gamma\vec{C}.$$

Now take the scalar product with $\vec{A}$

$$\vec{A}\cdot[\vec{A}\times(\vec{B}\times\vec{C})] = 0 = \beta\,\vec{A}\cdot\vec{B} + \gamma\,\vec{A}\cdot\vec{C}$$

since these are $\perp$

So if $\beta = \alpha(\vec{A}\cdot\vec{C})$ then $\gamma = -\alpha(\vec{A}\cdot\vec{B})$

$$\Longrightarrow \vec{A}\times(\vec{B}\times\vec{C}) = \alpha\left[\vec{B}(\vec{A}\cdot\vec{C}) - \vec{C}(\vec{A}\cdot\vec{B})\right]$$

To find α, ~~we~~ go explicitly to the above Cartesian coordinates

I.D.3.a.) $\vec{A} = A_x \hat{i} + A_y \hat{j} + A_z \hat{k}$

$\vec{B} = B_x \hat{i}$, $\vec{C} = C_x \hat{i} + C_y \hat{j}$

$\vec{B} \times \vec{C} = B_x C_y \hat{i} \times \hat{j} = B_x C_y \hat{k}$

$$\boxed{\vec{A} \times (\vec{B} \times \vec{C}) = -A_x B_x C_y \hat{j} + A_y B_x C_y \hat{i}}$$

On the other hand

$\vec{B}(\vec{A} \cdot \vec{C}) = B_x (A_x C_x + A_y C_y) \hat{i}$

$-\vec{C}(\vec{A} \cdot \vec{B}) = -A_x B_x C_x \hat{i} - A_x B_x C_y \hat{j}$

$$\boxed{\vec{B}(\vec{A} \cdot \vec{C}) - \vec{C}(\vec{A} \cdot \vec{B}) = A_y B_x C_y \hat{i} - A_x B_x C_y \hat{j}}$$

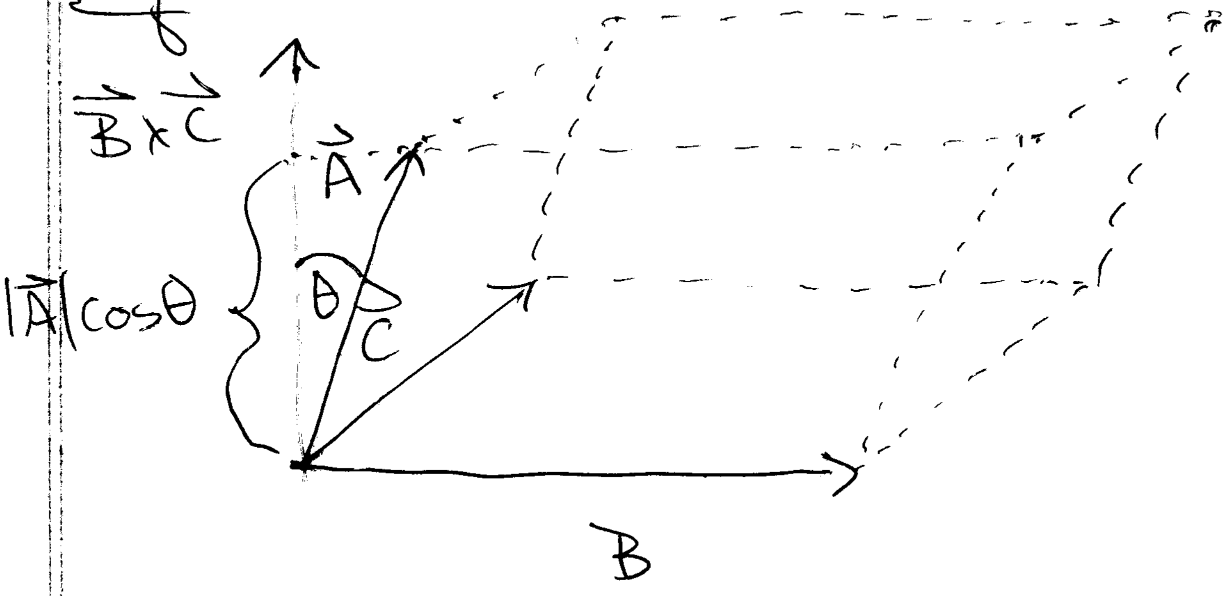
Thus $= \vec{A} \times (\vec{B} \times \vec{C})$

$$\boxed{\alpha = 1}$$

I.D.3.b.) Scalar Triple Product:

$$\vec{A}\cdot(\vec{B}\times\vec{C}) = \vec{B}\cdot(\vec{C}\times\vec{A}) = \vec{C}\cdot(\vec{A}\times\vec{B})$$

Proof:



Area of Base $= |\vec{B}\times\vec{C}|$

Height $= |\vec{A}|\cos\theta$

Volume $= |\vec{B}\times\vec{C}|\,|\vec{A}|\cos\theta$

$= \vec{A}\cdot(\vec{B}\times\vec{C})$

Thus Volume of parallelepiped with $\vec{A}, \vec{B}, \vec{C}$ as edges is given by the scalar triple product

$$\text{Vol.} = \vec{A}\cdot(\vec{B}\times\vec{C})$$

I.D.3.b.) The volume of the parallelopiped is the same if we take any other face as the base

$$\text{Volume} = \vec{B} \cdot (\vec{C} \times \vec{A})$$
$$= \vec{C} \cdot (\vec{A} \times \vec{B})$$

Thus

$$\boxed{\vec{A} \cdot (\vec{B} \times \vec{C}) = \vec{B} \cdot (\vec{C} \times \vec{A}) = \vec{C} \cdot (\vec{A} \times \vec{B}).}$$

This all follows from the determinant rule in components

$$\vec{A} \cdot (\vec{B} \times \vec{C}) = \begin{vmatrix} A_x & A_y & A_z \\ B_x & B_y & B_z \\ C_x & C_y & C_z \end{vmatrix}$$

The determinant changes sign when two rows are interchanged.

<u>Note</u>: The dot and cross can be interchanged without affecting the result

$$\vec{A} \cdot (\vec{B} \times \vec{C}) = (\vec{A} \times \vec{B}) \cdot \vec{C} = \vec{C} \cdot (\vec{A} \times \vec{B})$$

(the parantheses are necessary).