MATH. PHYS. REVIEW

Mathematical Preliminaries

Vector Algebra, Vector Calculus, Tensors and Generalized Functions

Due to the homogeneity and isotropy of space and the homogeneity of time the laws of physics have the same form in all inertial frames (covariance). Hence these laws are formulated in terms of tensorial equations which manifestly preserved this covariance. Tensors are defined to have specific transformation laws in going from inertial frame to inertial frame. The simplest tensor is a scalar, a quantity that is the same in all frames; it is an invariant. The mass of a particle is an example. The next simplest quantity is a vector, a tensor of rank 1. Vectors have magnitude and direction. Their components relative to different interial frames are related by rotations or velocity boosts. Newton's laws $\vec{F} = m\vec{a}$ is a vectorial equation, it is covariant under a change of frames. Thus vector (as well as tensor) manipulation are important tools in the study of these physical laws and quantities. Indeed many concepts from vector calculus such as surface integrals, line integrals will be needed. Derivatives like the gradient of a scalar function or the curl or the divergence of a vector function as well as integral theorems like Gauss' and Stokes' theorems will be used in formulating physical principles. Thus let's briefly review these vector algebra and vector calculus concepts as well as extend them to tensors and generalize the concept of functions to distributions.

I.) Vector Algebra:

Many of the quantities of physics can be divided into two groups: Scalars and vectors. Each has their own properties and algebra.

We will motivate the vector algebra rules from geometric considerations. This has the conceptual advantage that ab initio they are relationships that are independent of the specific inertial frame one might eventually use for component expressions.

First to define scalar quantities and their algebra.

I.A) <u>Scalar quantities</u>: magnitude only

(simplest) e.g. mass, length, temperature, electric charge. The algebra is that of ordinary real numbers, that is scalars have the ordinary operations of addition and multiplication as real numbers. These operations obey properties known as scalar algebra laws.

I.A) Scalar Algebra Laws:

a) Associative Laws

$$a+(b+c) = (a+b)+c = a+b+c$$
$$a\times(b\times c) = (a\times b)\times c = abc$$

b) Commutative Laws

$$a+b = b+a$$
$$a\times b = b\times a$$

c) Distributive Laws

$$a\times(b+c) = a\times b + a\times c$$
$$(a+b)\times c = a\times c + b\times c$$

I.B) Vector quantities: have both magnitude and direction eg. position, force, velocity, electric field. The vector algebra, that is the rules of manipulation of vectors, addition and multiplication, is quite different from that of scalars.

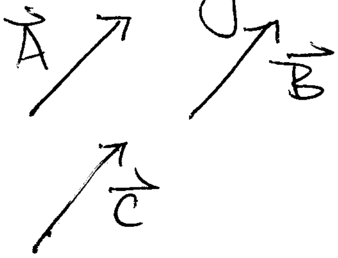
We will motivate these vector algebra rules from geometric considerations.

I.B.) Vectors are represented by arrows. Vector $\vec{A}$ has magnitude given by the length of the arrow

<u>length</u> = magnitude of $\vec{A} \equiv |\vec{A}|$

and <u>orientation</u> of the arrow = <u>direction</u> of $\vec{A}$.

1) Two vectors are equal if equal in magnitude and direction (i.e. ∥)

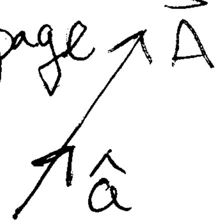


$$\vec{A} = \vec{B} = \vec{C}$$

2) <u>Unit Vector</u>, has unit magnitude and is dimensionless. A unit vector $\hat{a}$, can be constructed from vector $\vec{A}$ by

$$\hat{a} = \frac{\vec{A}}{|\vec{A}|} \quad ; \quad \vec{A} = |\vec{A}|\,\hat{a}$$

insert after scalar multiplication on next page



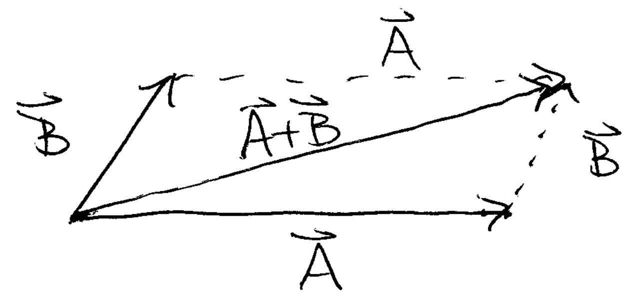
So $|\hat{a}| = 1$.

I.B.) 3) Zero or Null vector: has zero length and indeterminate direction.

I.B.) 5.) Addition of Vectors: Consider vectors $\vec{A}, \vec{B}$.

The sum $\vec{A}+\vec{B}$ is formed by using the parallelogram rule. Draw $\vec{B}$ from the tip of $\vec{A}$; the vector joining the tail of $\vec{A}$ to the head of $\vec{B}$ is defined as $\vec{A}+\vec{B}$

(interchange order)

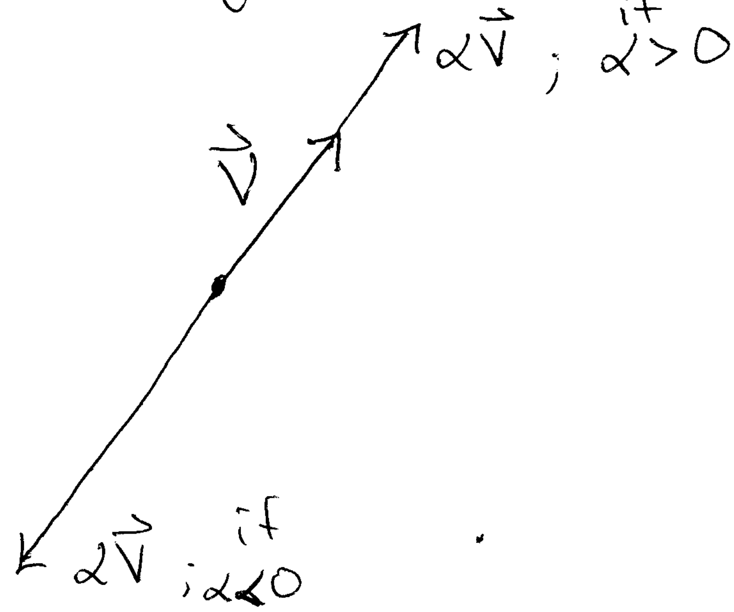


I.B.) 4.) Multiplication by a Scalar: let

α = a scalar, (ex. real number) and $\vec{V}$ be a vector. Then we define the vector $\alpha\vec{V}$ to be the vector with magnitude $|\alpha||\vec{V}|$ and direction

$$\begin{cases} \text{same as } \vec{V} & \text{if } \alpha > 0 \\ \text{opposite } \vec{V} & \text{if } \alpha < 0 \\ \vec{0} & \text{if } \alpha = 0 \end{cases}$$

I.B.4.) we define $\alpha\vec{V} = \vec{V}_\alpha$



I.B.6.) For α, β = scalars and $\vec{V}, \vec{W}$ vectors it follows that

i) Associative Law of Multiplication

$$(\alpha\beta)\vec{V} = \alpha(\beta\vec{V})$$

a) examples

$$\alpha(-\vec{V}) = (-\alpha)\vec{V}$$
$$-(-\vec{V}) = +\vec{V}$$

ii) Distributive Law

$$(\alpha+\beta)\vec{V} = \alpha\vec{V} + \beta\vec{V} \quad \text{(parallel addition)}$$

I.B.6) iii) Distributive Law

$$\alpha(\vec{V}+\vec{W}) = \alpha\vec{V} + \alpha\vec{W} .$$

(i.e. double sides of a parallelogram you double diagonal, etc.)

Further Properties of Vector Addition

I.B.6) iv) Commutivity for addition

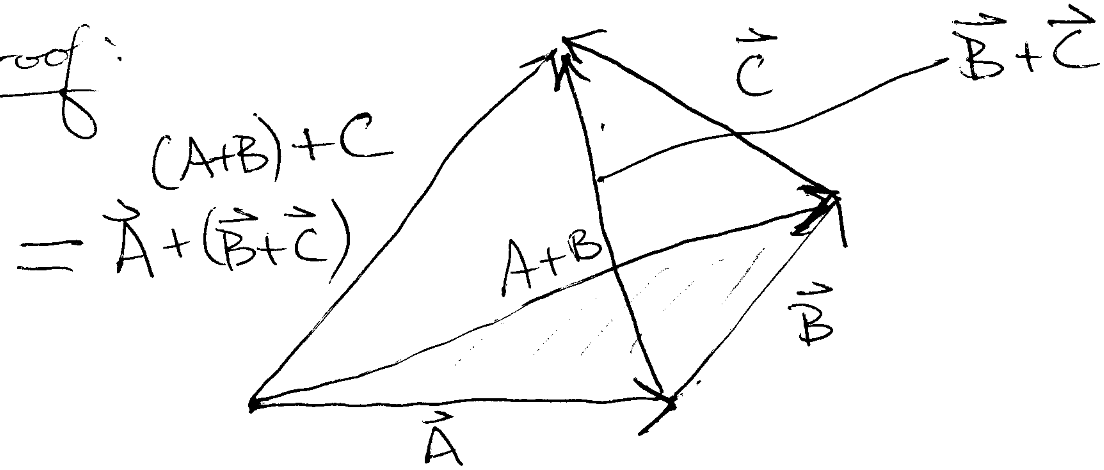
$$\vec{V}+\vec{W} = \vec{W}+\vec{V}$$

(Same diagonal of parallelogram)

v.) Associativity for addition

$$\vec{A}+(\vec{B}+\vec{C}) = (\vec{A}+\vec{B})+\vec{C}$$

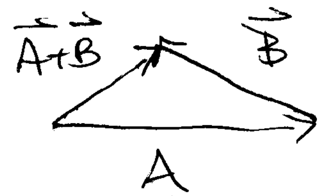
Proof:



I.B.6) vi) Triangle Inequality

$$|\vec{A}+\vec{B}| \leq |\vec{A}| + |\vec{B}|$$

This follows from the sum of any 2 sides of a triangle is greater than or equal to the third side



I.B.6) vii) $\vec{O}$ is the additive identity

$$\vec{A} + \vec{O} = \vec{A}$$

since $\vec{O}$ has tail and head the same.

I.) C.) Cartesian Coordinate System (components)

Definition: The vectors $\vec{V}_1, \vec{V}_2, \ldots, \vec{V}_N$ are linearly dependent if we can find numbers $\alpha_1, \ldots, \alpha_N$, not all zero, such that $\alpha_1\vec{V}_1 + \alpha_2\vec{V}_2 + \cdots + \alpha_N\vec{V}_N = 0$.