A Quadratic Method for Nonlinear Model Order Reduction

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ABSTRACT

In order to simulate and optimize efficiently systems which include micromachined devices, designers need dynamically accurate macromodels for the those devices. Although it is possible to develop such macromodels by hand, it would be vastly more efficient if it were possible to automatically derive such macromodels directly from physical coupled-domain simulation. Although such automatic techniques exist if the problem is linear, most micromachined devices are at least mildly nonlinear and new techniques must be developed. In this paper we present a quadratic reduction method which makes use of the Krylov subspace generated from linearized analysis. The result is a reduced-order model with a quadratic nonlinearity. Results on using the method for a nonlinear resistor network show that the nonlinear approach is much more accurate than using a linearized approach alone.

Keywords: model-order reduction

Introduction

Integrated circuit designers have sophisticated tools for handling electrical problems associated with interconnect and packaging. They describe the geometry of the interconnect structure to a three-dimensional electromagnetic analysis program, and that program automatically generates dynamically accurate macromodels of the input-output behavior of the structure. Then, the designer can include this macromodel in a circuit-level simulation of the entire system [1]. Designers of novel micromachined devices need just such a capability, so they can simulate systems which include their new devices. However, unlike interconnect and packaging, most micromachined devices are nonlinear, and extracting dynamically accurate nonlinear macromodels from simulation is an open problem. For this reason, there has been much current interest in developing nonlinear model-order reduction strategies [2]–[4].

In [5], a dynamically accurate macromodel for an electrostatically deformed beam was generated using linearization and an Arnoldi type state projection. In this paper we present a quadratic reduction method which makes use of the same projection space, but uses the method to reduce a nonlinear system to a small *nonlinear* system with quadratic nonlinearity. We begin below with a short review of Arnoldi model reduction for linear systems and then describe the quadratic reduction algorithm. Finally, results are presented for a nonlinear resistor network.

Background on Arnoldi Methods

Consider a single input, single output (SISO) linear system of the form

$$\begin{aligned} \boldsymbol{A}\,\dot{\boldsymbol{x}} &= \boldsymbol{x} + \boldsymbol{b}u \\ y &= \boldsymbol{c}^T\boldsymbol{x}. \end{aligned} \tag{1}$$

where $\boldsymbol{A}$ is a $n \times n$ matrix, and $\boldsymbol{b}$ and $\boldsymbol{c}$ are n-length vectors.

From (1), the input-output transfer function, $H(s)$, is given by

$$H(s) = \frac{y(s)}{u(s)} = -\boldsymbol{c}^T\left(\boldsymbol{I} - s\boldsymbol{A}\right)^{-1}\boldsymbol{b} \tag{2}$$

where s is the Laplace transform variable.

The transfer function $H(s)$ can be expanded in a Taylor series as

$$H(s) = \sum_{i=0}^{\infty} m_i s^i = \sum_{i=0}^{\infty} -\boldsymbol{c}^T\boldsymbol{A}^{-(i+1)}\boldsymbol{b}s^i \tag{3}$$

where m_i, the coefficient of the i^{th} term in the Taylor series, is known as the i^{th} moment of the transfer function.

A reduced-order model for (1) is the SISO system

$$\begin{aligned} \boldsymbol{A}_q\,\dot{\boldsymbol{x}}_q &= \boldsymbol{x}_q + \boldsymbol{b}_q u \\ \hat{y} &= \boldsymbol{c}_q^T\boldsymbol{x}_q \end{aligned} \tag{4}$$

where $\boldsymbol{x}_q, \boldsymbol{b}_q, \boldsymbol{c}_q \in \mathbf{R}^q$, $\boldsymbol{A}_q \in \mathbf{R}^{q\times q}$ and q is presumably much smaller than n. The model-order reduction problem is then finding the smallest $\boldsymbol{A}_q$, $\boldsymbol{b}_q$ and $\boldsymbol{c}_q$ such that

$$\hat{H}(s) = \frac{\hat{y}(s)}{u(s)} = -\boldsymbol{c_q}^T\left(\boldsymbol{I} - s\boldsymbol{A_q}\right)^{-1}\boldsymbol{b_q} \tag{5}$$

approximates $H(s) = \frac{y(s)}{u(s)}$ with sufficient accuracy.

A commonly used approach for generating reduced-order models is based on the Arnoldi algorithm for robustly generating an orthonormal basis for the Krylov subspace given by

$$\mathcal{K}_k(\boldsymbol{A}, \boldsymbol{b}) = span\{\boldsymbol{b}, \boldsymbol{A}\boldsymbol{b}, \boldsymbol{A}^2\boldsymbol{b}, \cdots, \boldsymbol{A}^{k-1}\boldsymbol{b}\}. \tag{6}$$

The basic idea of the Arnoldi approach is to generate a k^{th} order orthogonalized Krylov subspace from a $(k-1)^{th}$

order orthogonalized Krylov subspace. First, the last vector in the $(k-1)^{th}$-order orthogonalized subspace is multiplied by A, and then the resulting vector is orthogonalized with respect to the previous $k-1$ vectors.

After q steps, the Arnoldi algorithm returns a set of q orthonormal vectors, as the columns of the matrix $\boldsymbol{V}_q \in \mathbf{R}^{n\times q}$, and a $q \times q$ upper Hessenberg (tridiagonal plus upper triangular) matrix $\boldsymbol{H}_q$ whose entries are the scalars $h_{i,j}$ generated by the Arnoldi algorithm. These two matrices satisfy the following relationship:

$$\boldsymbol{A}\ \boldsymbol{V}_q = \boldsymbol{V}_q\ \boldsymbol{H}_q + h_{q+1,q}\ \boldsymbol{v}_{q+1}\ \boldsymbol{e}_q^T \tag{7}$$

where $\boldsymbol{e}_q$ is the q^{th} unit vector in $\mathbf{R}^q$. From (7), it can easily be seen that after q steps of an Arnoldi process, for $k < q$,

$$\boldsymbol{A}^k\ \boldsymbol{b} = ||\boldsymbol{b}||_2\ \boldsymbol{A}^k\ \boldsymbol{V}_q\ \boldsymbol{e}_1 = ||\boldsymbol{b}||_2\ \boldsymbol{V}_q\ \boldsymbol{H}_q^k\ \boldsymbol{e}_1. \tag{8}$$

With this relation, the moments (3) can be related to $\boldsymbol{H}_q$ by

$$m_k = \boldsymbol{c}^T\ \boldsymbol{A}^k\ \boldsymbol{b} = \underbrace{||\boldsymbol{b}||_2\ \boldsymbol{c}^T\ \boldsymbol{V}_q}_{\boldsymbol{c}_q^T}\ \underbrace{\boldsymbol{H}_q^k}_{\boldsymbol{A}_q^k}\ \underbrace{\boldsymbol{e}_1}_{\boldsymbol{b}_q} \tag{9}$$

and so, by analogy with (3), the q^{th} order Arnoldi-based approximation to $H(s)$ can be written as

$$H(s) \approx \hat{H}(s)||\boldsymbol{b}||_2\ \boldsymbol{c}^T\ \boldsymbol{V}_q\ (\boldsymbol{I} - s\boldsymbol{H}_q)^{-1}\ \boldsymbol{e}_1 \tag{10}$$

corresponding to the state-space realization $\boldsymbol{A}_q = \boldsymbol{H}_q$, $\boldsymbol{b}_q = \boldsymbol{e}_1$, and $\boldsymbol{c}_q = ||\boldsymbol{b}||_2\ \boldsymbol{V}_q^T\ \boldsymbol{c}$.

Another way of viewing the Arnoldi reduction is to consider that V_q introduces a nonsquare change of variables of the form

$$x = V_q x_q. \tag{11}$$

Substituting the change of variables in (1) and multiplying through by $\boldsymbol{V}_q^T$ yields

$$\begin{array}{rcl} \boldsymbol{V}_q^T\boldsymbol{A}\boldsymbol{V}_q\ \dot{\boldsymbol{x}} & = & \boldsymbol{V}_q^T\boldsymbol{x}_q\ +\ \boldsymbol{V}_q^T\boldsymbol{b}u \\ \hat{y} & = & \boldsymbol{c}^T\boldsymbol{V}_q^T\boldsymbol{x}_q. \end{array} \tag{12}$$

Matching corresponding terms in (1) and (12) results in exactly the same reduced-order model as matching corresponding terms in the transfer functions (2) and (5). The change of variables point of view, however, extends more readily to the nonlinear case.

Nonlinear Model Order Reduction

To briefly describe the method, consider a nonlinear system

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}) + \mathbf{b}u(t) \quad y = \mathbf{c}^{\mathbf{T}}\mathbf{x} \tag{13}$$

were $\mathbf{x}$ is an n-length vector, $\mathbf{f}$ is a nonlinear vector function. The above system with a nonlinear state equation is referred to as the "original" system which will be reduced to a much smaller system. Here $u(t)$ is the input of the system and $y(t)$ the output. The reduced system is expect to preserve the input-output behavior.

Taylor expanding the function $\mathbf{f}$ to second order about the origin yeilds a quadratic approximation of the above system

$$\dot{\mathbf{x}} = \mathbf{J_f}\mathbf{x} + \mathbf{x}^T\mathbf{W}\mathbf{x} + \mathbf{b}u(t) \quad y = \mathbf{c}^T\mathbf{x} \tag{14}$$

where $\mathbf{J_f}_{i,j} = \frac{\partial f_i}{\partial x_j}$ is the Jacobian of f evaluated about zero, and $\mathbf{W}$ is an $N \times N \times N$ Hessian tensor whose entrees are given by

$$\mathbf{W}_{i,j,k} = \frac{\partial^2 f_i}{\partial x_j \partial x_k}. \tag{15}$$

Assuming $\mathbf{J_f}$ is *nonsingular*, the above quadratic system can be put in a form similar to that used in the linear Arnoldi algorithm above. Let $A = J_f^{-1}$, then multiplying (14) by A results in

$$\mathbf{A}\dot{\mathbf{x}} = \mathbf{x} + \mathbf{A}\mathbf{x}^T\mathbf{W}\mathbf{x} + \mathbf{A}\mathbf{b}u(t) \quad y = \mathbf{c}^{\mathbf{T}}\mathbf{x}. \tag{16}$$

The Arnoldi method can be used to construct an orthogonal basis for the Krylov subspace

$$Span\{\mathbf{A}\mathbf{b}, \mathbf{A}^2\mathbf{b}, \cdots, \mathbf{A}^q\mathbf{b}\}$$

where q will be the size of the reduced system. The Arnoldi process generates $\mathbf{V}$, an $n \times q$ orthonormal matrix whose columns span the Krylov subspace, and $\mathbf{H} = \mathbf{V}^T\mathbf{A}\mathbf{V}$.

Following the linear model order reduction approach above, consider introducing the nonsquare change of variables

$$\mathbf{x} = \mathbf{V}\mathbf{z}. \tag{17}$$

Using (17) in (16),

$$\begin{array}{rcl} \mathbf{A}\mathbf{V}\dot{\mathbf{z}} & = & \mathbf{V}\mathbf{z} + \mathbf{A}\mathbf{z}^T\mathbf{V}^T\mathbf{W}\mathbf{V}\mathbf{z} + \mathbf{b}_1 u(t) \\ y & = & \mathbf{c}^{\mathbf{T}}\mathbf{V}\mathbf{z}. \end{array}$$

Substituting using the definition of the matrix $\mathbf{H}$,

$$\begin{array}{rcl} \mathbf{H}\dot{\mathbf{z}} & = & \mathbf{V}^{\mathbf{T}}\mathbf{A}\mathbf{V}\dot{\mathbf{z}} = \mathbf{z} + \mathbf{V}^T\mathbf{A}\mathbf{z}^T\mathbf{V}^T\mathbf{W}\mathbf{V}\mathbf{z} + \mathbf{V}^T\mathbf{b}_1 u(t) \\ y & = & \mathbf{c}^{\mathbf{T}}\mathbf{V}\mathbf{z}. \end{array}$$

The system can be returned to the normal form of (14) by multiplying though by $\mathbf{H}^{-1}$,

$$\begin{array}{rcl} \dot{\mathbf{z}} & = & \mathbf{H}^{-1}\mathbf{z} + \mathbf{H}^{-1}\mathbf{V}^T\mathbf{A}\mathbf{z}^T\mathbf{V}^T\mathbf{W}\mathbf{V}\mathbf{z} + \mathbf{H}^{-1}\mathbf{V}^T\mathbf{b}_1 u(t) \\ y & = & \mathbf{c}^{\mathbf{T}}\mathbf{V}\mathbf{z}. \end{array}$$

Now denote

$$\hat{\mathbf{J}} = \mathbf{H}^{-1} \tag{18}$$

$$\hat{\mathbf{b}} = \mathbf{H}^{-1}\mathbf{V}^T\mathbf{b} \tag{19}$$

$$\hat{\mathbf{c}} = \mathbf{V}^T\mathbf{c}. \tag{20}$$

The term

$$\mathbf{H}^{-1}\mathbf{V}^T\mathbf{A}\mathbf{z}^T\mathbf{V}^T\mathbf{W}\mathbf{V}\mathbf{z}$$

is quadratic in z, and so can be written in the form $\mathbf{z}^T\hat{\mathbf{W}}\mathbf{z}$ for some $\hat{\mathbf{W}}$. Then, (18) can be reduced to a quadratic system of the form

$$\begin{aligned}\dot{\mathbf{z}} &= \hat{\mathbf{J}}\mathbf{z} + \mathbf{z}^T\hat{\mathbf{W}}\mathbf{z} + \hat{\mathbf{b}}u(t)\\ \hat{y} &= \hat{\mathbf{c}}^T\mathbf{z}\end{aligned}$$

The key point is that the Arnoldi projection was used to reduce the large quadratic tensor to a small quadratic tensor.

Preliminary Results

To demonstrate the method, consider the capacitor and nonlinear reistor circuit example shown in Figure 1. The nonlinear resistors (a diode in parallel with a unit resistor) have the constitutive relation $i(v) = (\exp(40v) - 1) + v$ and the capacitors have unit capacitance. The input is a current source entering node 1, and the output is the voltage at node 1. The number of nodes in the original system is $N = 100$. The quadratic method was used to reduce the system to $q = 10$ and Figure 2 and 3 compare the outputs of the reduced systems with those original and linearization systems in response to two chosen inputs. Although our implementation is in Matlab, and so computational comparisons can only be used to determine trends, it is interesting to note that the dynamic simulation of the 10th order model is much faster than the integration of the original system (See Table 1).

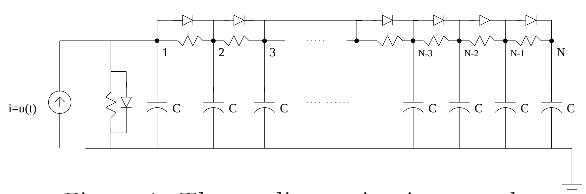

Figure 1: The nonlinear circuit example

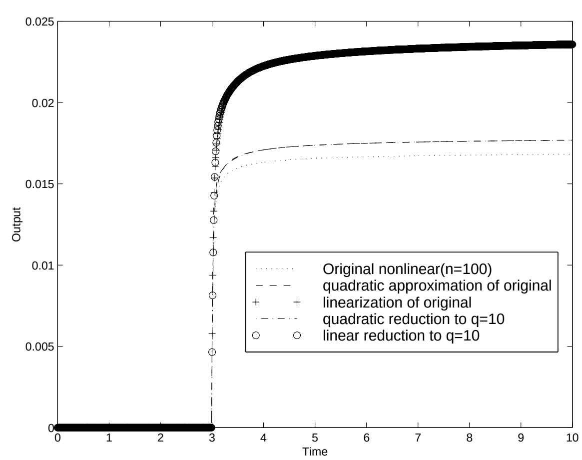

Figure 2: Comparison of the original nonlinear system(size 100) with the reduced systems generated by quadratic reduction and by linearization to size 10.The response output is for the step source and here we also plot the original quadratic approximation and linearization systems for reference

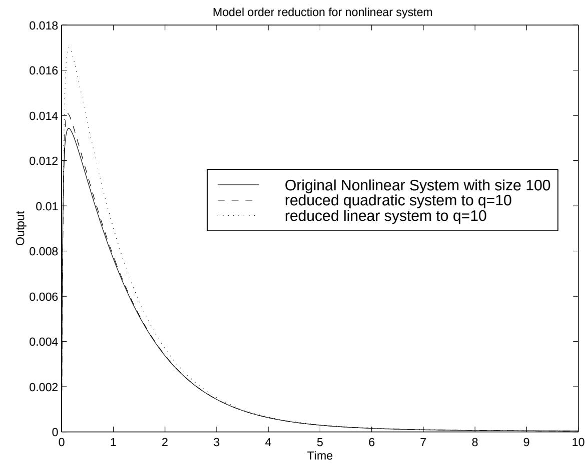

Figure 3: Comparison between the same original nonlinear system with its quadratic and linear reductions for an input source $u = e^{-t}$

CPU time (in sec)	Full (size 100)	reduced (quad) system(size 10)	reduced (lin) system(size 10)
$u =$ step	98.3	6.74	1.60
$u = e^{-t}$	115.3	6.77	1.60

Table 1: Comparison of computation time to integrating original nonlinear system and its quadratic and linear reduced systems

Conclusions and Acknowledgements

In this paper we presented a quadratic reduction method which makes use of the Krylov subspace generated from linearized analysis. The result is a reduced-order model with a quadratic nonlinearity. Results on using the method for a nonlinear resistor network show that the nonlinear approach is much more accurate than using a linearized approach alone.

Note that the reduced quadratic system can be derived without explicitly computing the full quadratic approximations of the original system [6]. Also, it is possible to improve accuracy by extending the above method to include a third order approximation, but the cost of the reduced-order model increases like q^4 and even a tenth order model has 10,000 coefficients. Finally, the notation used herein and in [6] is cumbersome, and the choice of projection space is ad-hoc. A much cleaner presentation using Kronecker products and a theoretically sounder approach for selecting the projection space is given in [4].

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