Optical theorem formulation of low-energy nuclear reactions

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We present a detailed description of a recently proposed optical theorem formulation of nonresonant low-energy nuclear reactions between two charged nuclei. Based on the optical theorem formulation, we obtain an analytic formula for the reaction cross section $\sigma(E)$ which exhibits explicitly the energy and charge dependences of $\sigma(E)$. The formula may provide a physical understanding of the anomalous low-energy enhancement of $\sigma(E)$ observed in sub-barrier heavy-ion fusions and also in light nuclei fusions relevant for primordial nucleosynthesis and stellar evolution. As examples of its application, the new formulation is used to analyze astrophysical S factors for $^{7}\text{Li}(p,\alpha)^{4}\text{He}$, $^{6}\text{Li}(d,\alpha)^{4}\text{He}$, and $^{6}\text{Li}(p,\alpha)^{3}\text{He}$ reactions. [S0556-2813(97)03602-9]

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I. INTRODUCTION

Primordial big-bang nucleosynthesis and the nucleosynthesis involved in stellar evolution are qualitatively understood, but there are many questions which need to be answered before we can reach a more quantitative understanding of the primordial nucleosynthesis and stellar evolution. In particular, there remain uncertainties due to the unknown rates and cross sections $\sigma(E)$ for many nuclear reactions involved in nuclear astrophysics calculations. Since $\sigma(E)$ at energies ($\lesssim$ a few keV) relevant to primordial and stellar nucleosyntheses cannot be measured in the laboratory, they are extracted from laboratory measurements of $\sigma(E)$ at higher energies by an extrapolation procedure based on nuclear theory. However, the energy dependence of the nuclear reaction cross section $\sigma(E)$ cannot be obtained rigorously from first principles since many-nucleon scattering problems cannot be solved exactly even if the nucleon-nucleon force is given. Therefore, one must rely on physically reasonable nuclear reaction models, such as optical potential models (OPM's) [1,2], cluster models based on the resonating group method (RGM) [3–6] or on the generator coordinate method (GCM) [7], and the P-matrix method [8]. Recently, we introduced an alternative theoretical formulation [9] of nuclear reactions which is based on the optical theorem. Our new formulation can be applied to both resonance and nonresonant nuclear reactions. In this paper, we present a more detailed description of our optical theorem formulation for the nonresonant case.

For nonresonant nuclear reactions, it is customary to extract the astrophysical S factor $S(E)$ from the experimentally measured $\sigma(E)$ using the formula

$$\sigma_G(E) = \frac{S(E)}{E} e^{-2\pi\eta(E)}, \tag{1}$$

where $\eta(E) = Z_a Z_b e^2/\hbar v$ is the Sommerfeld parameter and $e^{-2\pi\eta(E)}$ is the Gamow factor representing the probability of bringing two charged nuclei to zero separation distance. $S(E)$ is expected to be a slowly varying function of E. Recent results for $\sigma(E)$ from laboratory beam experiments for nuclear reactions involving light nuclei at low energies (>3 keV) show that the extracted $S(E)$ increased toward lower energies instead of being a constant extrapolated from higher-energy data, indicating the possibility of the importance of electron screening. However, recent theoretical calculations [10,11] of the electron screening effect yield limiting values which are much smaller (by $\sim 1/2$) than those extracted from the experimental data for the reactions $^{3}\text{He}(d,p)^{4}\text{He}$ [12,13], $^{6}\text{Li}(p,\alpha)^{3}\text{He}$, $^{6}\text{Li}(d,\alpha)^{4}\text{He}$, $^{7}\text{Li}(p,\alpha)^{4}\text{He}$ [14], $^{10}\text{B}(p,\alpha)^{7}\text{Be}$, and $^{11}\text{B}(p,\alpha)^{8}\text{Be}$ [15]. This discrepancy between the experimental data and the theoretical estimate for the electron screening effect is not well understood at present.

In this paper, as examples of the application of the optical theorem formulation, we present an analysis of the low-energy fusion reactions $^{6}\text{Li}(p,\alpha)^{3}\text{He}$, $^{6}\text{Li}(d,\alpha)^{4}\text{He}$, and $^{7}\text{Li}(p,\alpha)^{4}\text{He}$ [14]. The investigation of these reactions is important in two aspects: (i) These reactions are dominant depletion processes for ^{6}Li and ^{7}Li, and their improved reaction rates could provide an explanation for the extremely low lithium abundance and also could improve the present theories of spallative and big-bang nucleosynthesis generation of light elements [16], and the fusion reactions $^{6}\text{Li}(p,\alpha)^{3}\text{He}$ and $^{6}\text{Li}(d,\alpha)^{4}\text{He}$ are relatively clean energy sources as well as highly efficient mechanisms for energy generation among exotic nuclear fuel elements (Li, Be, and B) [17]. Reaction rates for $^{6}\text{Li}(p,\alpha)^{3}\text{He}$ and $^{6}\text{Li}(d,\alpha)^{4}\text{He}$ are approximately 10^{-4} and 10^{-3}, respectively, of that for $T(d,n)^{4}\text{He}$ at an incident ion energy of ~ 25 keV. These reaction rates are relatively large compared to those for other exotic nuclear fuels.

In Sec. II, we give a derivation of the low-energy "partial-wave" optical theorem [9], which is different from the conventional optical theorem. The relationship between

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the total reaction cross section and the T matrix is also given in the partial-wave form. In Sec. III, using a separable form of the T matrix with several parameters, we obtain an analytical formula for the reaction cross section, which exhibits explicitly the energy and charge dependences of the reaction cross section. In Sec. IV, the analytic formula for the cross section derived in Sec. III is applied to an analysis of experimental data for the fusion reactions $^6\text{Li}(p,\alpha)^3\text{He}$, $^6\text{Li}(d,\alpha)^4\text{He}$, and $^7\text{Li}(p,\alpha)^4\text{He}$. A summary and conclusions are given in Sec. V.

II. OPTICAL THEOREM FORMULATION

In this section, we introduce a low-energy ''partial-wave'' optical theorem and use it to develop the optical theorem formulation of low-energy nuclear reactions.

The conventional optical theorem first introduced by Feenberg [18] is given by

$$\sigma_t = \frac{4\pi}{k} \, \text{Im} \, f(0), \qquad (2)$$

where σ_t is the total cross section and $f(0)$ is the elastic scattering amplitude in the forward direction ($\theta=0$). There are many other forms of the ''optical theorem,'' all of which are physically related to the interference [19]. A generalized optical theorem for the case of the Coulomb interaction plus nuclear forces was developed and improved by Marty [20] (see also [21]).

To avoid complications associated with the singularity of the forward Coulomb scattering amplitude $f^c(0)$ as in the case of the conventional optical theorem, for two-potential scattering involving two charged nuclei, we describe a different formulation based on a partial-wave optical theorem involving angle-integrated and/or angle-independent quantities in the following.

For the elastic scattering involving the Coulomb interaction and nuclear potential, the scattering amplitude can be written as a sum of two amplitudes:

$$f(\theta) = f^c(\theta) + \tilde{f}(\theta), \qquad (3)$$

where $f^c(\theta)$ is the Coulomb amplitude and $\tilde{f}(\theta)$ is the remainder. $\tilde{f}(\theta)$ can be expanded in partial waves [22] as

$$\tilde{f}(\theta) = \sum_l (2l+1) e^{2i\delta_l^c} f_l^{N(\text{el})} P_l(\cos\theta), \qquad (4)$$

where δ_l^c is the Coulomb phase shift, $f_l^{N(\text{el})} = (S_l^N - 1)/2ik$, and S_l^N is the lth partial wave S matrix for the nuclear part. The partial-wave expansion of the nuclear elastic cross section $\sigma^{N(\text{el})}$ is given by $\sigma^{N(\text{el})} = \Sigma_l (2l+1) \sigma_l^{N(\text{el})}$ with $\sigma_l^{N(\text{el})} = (\pi/k^2)|S_l^N - 1|^2$.

For the reaction cross section $\sigma^{(r)}$, the partial-wave expansion is given by $\sigma^{(r)} = \Sigma_l (2l+1) \sigma_l^{(r)}$, where $\sigma_l^{(r)} = \pi(1 - |S_l^N|^2)/k^2$. Using the expressions of $\sigma_l^{(r)}$ and $\sigma_l^{N(\text{el})}$ derived above, we can write $\sigma_l^{(r)} + \sigma_l^{N(\text{el})} = 2\pi(1 - \text{Re} \, S_l^N)/k^2$. Combining this with $\text{Im} \, f_l^{N(\text{el})} = (1 - \text{Re} \, S_l^N)/2k$, we obtain the partial-wave optical theorem for the two-potential scattering case as

$$\text{Im} \, f_l^{N(\text{el})} = \frac{k}{4\pi} (\sigma_l^{(r)} + \sigma_l^{N(\text{el})}), \qquad (5)$$

which is a rigorous result.

For low energies, $f_l^{N(\text{el})} \propto e^{-2\pi\eta}/k$ and hence $\sigma_l^{N(\text{el})} = 4\pi|f_l^{N(\text{el})}|^2 \propto e^{-4\pi\eta}/k^2$. Since $\sigma_l^{(r)} \propto e^{-2\pi\eta}/k^2$, we have $\sigma_l^{(r)} \gg \sigma_l^{N(\text{el})}$ at low energies, and hence we can write Eq. (5) as

$$\text{Im} \, f_l^{N(\text{el})} \approx \frac{k}{4\pi} \sigma_l^{(r)}, \qquad (6)$$

which is still a rigorous result at low energies. We note that Eqs. (5) and (6) are nonradiative nuclear reactions and need to be modified for radiative nuclear reactions.

In terms of the partial wave T matrix, T_l, the elastic nuclear scattering amplitude $f_l^{N(\text{el})} = (S_l^N - 1)/2ik$ can be written as

$$f_l^{N(\text{el})}(E) = -\frac{2\mu}{\hbar^2 k^2} \langle \psi_l^c | T_l | \psi_l^c \rangle, \qquad (7)$$

where ψ_l^c is the lth partial-wave regular Coulomb function and μ is the reduced mass. Using the low-energy optical theorem Eq. (6) with Eq. (7), we obtain the lth partial-wave reaction cross section $\sigma_l(E) \, [=\sigma_l^{(r)}(E)]$ as

$$\sigma_l(E) \approx \frac{4\pi}{kE} \int_0^\infty \psi_l^c(r) U_l(r,r') \psi_l^c(r') dr \, dr', \qquad (8)$$

where $E = \hbar^2 k^2/2\mu$ and $U_l(r,r') = -\text{Im}\langle r|T_l|r'\rangle$ with T_l representing the lth partial-wave contribution of the T-matrix operator. The total reaction cross section $\sigma(E)$ is given by $\sigma(E) = \Sigma_l(2l+1)\sigma_l(E)$.

It is important to note that our optical theorem formulation of nuclear reactions, Eq. (6), can be applied to both nonresonant and resonance reactions using the T matrix given in Eq. (7). In the following, we consider only the nonresonant case.

Using the Feshbach projection method [23], the Schrödinger equation for a multichannel system can be written as

$$\left(-\frac{\hbar^2}{2\mu_i} \frac{d^2}{dr_i^2} + \frac{\hbar^2 l_i(l_i+1)}{2\mu_i r_i^2} + \frac{Z_a^{(i)} Z_b^{(i)} e^2}{r_i} \right) \psi_i(r_i)$$
$$+ \sum_j \int V_{ij}(r_i, r_j', E) \psi_j(r_j') dr_j' = (E + Q_i) \psi_i(r_i), \qquad (9)$$

where μ_i is the reduced mass, $Z_a^{(i)}$ and $Z_b^{(i)}$ are nuclear charges, and Q_i is the energy release for the ith channel. For the two-channel case, Eq. (9) is

$$\left(-\frac{\hbar^2}{2\mu_1} \frac{d^2}{dr_1^2} + \frac{\hbar^2 l_1(l_1+1)}{2\mu_1 r_1^2} + \frac{Z_a^{(1)} Z_b^{(1)} e^2}{r_1} \right) \psi_1(r_1)$$
$$+ \int V_{11}(r_1, r_1', E) \psi_1(r_1') dr_1'$$
$$+ \int V_{12}(r_1, r_2', E) \psi_2(r_2') dr_2' = E \psi_1(r_1) \qquad (10)$$

and

$$\left(-\frac{\hbar^2}{2\mu_2}\frac{d^2}{dr_2^2}+\frac{\hbar^2 l_2(l_2+1)}{2\mu_2 r_2^2}+\frac{Z_a^{(2)}Z_b^{(2)}e^2}{r_2}\right)\psi_2(r_2)$$

$$+\int V_{22}(r_2,r_2',E)\psi_2(r_2')dr_2'$$

$$+\int V_{21}(r_2,r_1',E)\psi_1(r_1')dr_1'=(E+Q)\psi_2(r_2).$$

$$(11)$$

To avoid complications associated with nonorthogonality of the basis states $|\phi_i\rangle$ [24], we describe a formulation based on the Faddeev-type differential equation [25] in Appendix A, which have a unique solution under some asymptotic boundary conditions only if binary (two-body) channels are open. The general properties of $V_{ij}(r_i,r_j',E)$ are

$$V_{ij}(r_i,r_j',E)=V_{ji}(r_j',r_i,E), \qquad (12)$$

and $V_{ij}(r_i,r_j',E)$ in this case are real functions. The symmetry in Eq. (12) is due to the fact that Eq. (9) is T invariant. $V_{ij}(r_i,r_j',E)$ is expected to be a slowly varying function of energy for low-energy nonresonant fusion reactions. We note that in the Faddeev-type RGM approximation, V_{ij}, are independent of energy, as shown in Appendix A.

To convert Eq. (11) into an integral equation, we introduce two linearly independent solutions $\phi(r_2)=X(r_2)$ and $\phi(r_2)=Y(r_2)$ of the equation

$$\left(-\frac{\hbar^2}{2\mu_2}\frac{d^2}{dr_2^2}+\frac{\hbar^2 l_2(l_2+1)}{2\mu_2 r_2^2}+\frac{Z_a^{(2)}Z_b^{(2)}e^2}{r_2}\right)\phi(r_2)$$

$$+\int V_{22}(r_2,r_2',E)\phi(r_2')dr_2'=(E+Q)\phi(r_2). \qquad (13)$$

For large r_2, solutions $X(r_2)$ and $Y(r_2)$ of Eq. (13) satisfy

$$X_2(r_2)=\sin\left(k_2 r_2-\frac{l_2\pi}{2}-\eta_2\ln 2k_2 r_2+\delta_{l_2}^c+\delta_{l_2}\right)$$

and

$$Y_2(r_2)=\cos\left(k_2 r_2-\frac{l_2\pi}{2}-\eta_2\ln 2k_2 r_2+\delta_{l_2}^c+\delta_{l_2}\right),$$

respectively, and hence the Wronskian $Y_2 X_2'-X_2 Y_2'=k_2$ with $k_2=[2\mu_2(E+Q)/\hbar^2]^{1/2}$. Using the above relations, we can rewrite Eq. (11) in the integral form

$$\psi_2(r_2)=-\frac{2\mu_2}{\hbar^2 k_2}\int G_2(r_2,r_2',E)V_{21}(r_2',r_1',E)$$

$$\times\psi_1(r_1')dr_2'dr_1', \qquad (14)$$

where

$$G_2(r_2,r_2',E)=\begin{cases} X_2(r_2)[Y_2(r_2')+iX_2(r_2')], & r_2<r_2', \\ [Y_2(r_2)+iX_2(r_2)]X_2(r_2'), & r_2'<r_2. \end{cases}$$

Substituting Eq. (14) into Eq. (10) yields

$$\left(-\frac{\hbar^2}{2\mu_1}\frac{d^2}{dr_1^2}+\frac{\hbar^2 l_1(l_1+1)}{2\mu_1 r_1^2}+\frac{Z_a^{(1)}Z_b^{(1)}e^2}{r_1}\right)\psi_1(r_1)$$

$$+\int V^s(r_1,r_1',E)\psi_1(r_1')dr_1'=E\psi_1(r_1), \qquad (15)$$

where an effective potential $V^s(r_1,r_1',E)$ is given by $V^s(r_1,r_1',E)=V_{\mathrm{Re}}^S(r_1,r_1',E)-iV_{\mathrm{Im}}^S(r_1,r_1',E)$, and

$$V_{\mathrm{Im}}^S(r_1,r_1',E)=\frac{2\mu_2}{\hbar^2 k_2}\int V_{12}(r_1,r_2',E)X_2(r_2')dr_2'$$

$$\times\int X_2(r_2'')V_{21}(r_2'',r_1',E)dr_2'', \qquad (16)$$

$$V_{\mathrm{Re}}^S(r,r',E)=V_{11}(r_1,r_1')-\frac{2\mu_2}{\hbar^2 k_2}\int V_{12}(r_1,r_2',E)$$

$$\times\mathrm{Re}\,G_2(r_2',r_2'',E)V_{21}(r'',r',E)dr_2'dr_2''. \qquad (17)$$

From Eqs. (12), (16), and (17), we can see that the imaginary part of $V^s(r_1,r_1',E)$ is separable and

$$V^s(r_1,r_1',E)=V^s(r_1',r_1,E). \qquad (18)$$

For $l=0$, the effective Hamiltonian in the elastic channel has the form

$$H_{\mathrm{el}}=-\frac{\hbar^2}{2\mu}\frac{d^2}{dr^2}+\frac{Z_a Z_b e^2}{r}+V_{\mathrm{Re}}^S-iV_{\mathrm{Im}}^S. \qquad (19)$$

The T matrix for Eq. (19) satisfies

$$T=(V_{\mathrm{Re}}^S-iV_{\mathrm{Im}}^S)+(V_{\mathrm{Re}}^S-iV_{\mathrm{Im}}^S)\frac{1}{E-H_0-V^C+i\epsilon}T$$

$$=(V_{\mathrm{Re}}^S-iV_{\mathrm{Im}}^S)+T\frac{1}{E-H_0-V^C+i\epsilon}(V_{\mathrm{Re}}^S-iV_{\mathrm{Im}}^S),$$

$$(20)$$

where

$$H_0=-\frac{\hbar^2}{2\mu}\frac{d^2}{dr^2}$$

and $V^C=Z_a Z_b e^2/r$. Using Eqs. (16), (17), and (18), it can be shown that the imaginary part of the T matrix is separable and symmetric (see Appendix B):

$$T(r_1,r_1',E)=T(r_1',r_1,E). \qquad (21)$$

III. REACTION CROSS-SECTION FORMULA

Since $U_0(r,r')$ in Eq. (8) is separable and symmetric for the two-channel case, we can write $U_0(r,r')$ as

$$U_0(r,r')=\lambda g(r)g(r'), \qquad (22)$$

where λ is expected to be a slowly varying function of energy for the nonresonant case. (For the case of resonance reactions, the energy dependence of λ can be parametrized

by the Breit-Wigner expression.) It should be noted that one advantage of Eq. (22) is that we can parametrize only one real function $g(r)$. However, to construct $g(r)$ exactly would be as difficult as solving the original two-channel equations.

For the $l=0$ case, the Coulomb wave function $\psi_0^c(r)$ is given by

$$\psi_0^c(r) = C_0(\eta) M_{i\eta,1/2}(2ikr)/2i, \qquad (23)$$

where $C_0^2(\eta) = 2\pi\eta/(e^{2\pi\eta} - 1)$ and $M_{i\eta,1/2}(2ikr)$ is the Whittaker function. The reaction cross section, Eq. (8), in the case of $g(r) = e^{-\beta r}$, can be written as

$$\sigma_0(E) = \frac{4\pi\lambda}{kE} \left(\int_0^\infty \psi_0^c(r) e^{-\beta r} dr \right)^2$$

$$= \frac{\pi\lambda}{kE} C_0^2(\eta) \left(\frac{1}{i} \int_0^\infty M_{i\eta,1/2}(2ikr) e^{-\beta r} dr \right)^2. \qquad (24)$$

The integral in Eq. (24) can be evaluated exactly:

$$\frac{1}{i} \int_0^\infty M_{i\eta,1/2}(2ikr) e^{-\beta r} dr$$

$$= \frac{1}{i} [(2ik)\Gamma(2)(\beta - ik)^{i\eta-1}(\beta + ik)^{-i\eta-1}]$$

$$= \frac{2k}{\beta^2 + k^2} e^{2\phi\eta}, \qquad (25)$$

where $\phi = \arctan(k/\beta)$. Thus, the reaction cross section $\sigma_0(E)$, in Eq. (24), is

$$\sigma_0(E) = \frac{4\pi^2\lambda}{R_B E} \frac{e^{4\phi\eta}}{(e^{2\pi\eta} - 1)(\beta^2 + k^2)^2}, \qquad (26)$$

which for the low-energy case reduces to

$$\sigma_0(E) = \frac{\widetilde{S}_0(E) e^{4\phi\eta}}{E} e^{-2\pi\eta}, \qquad (27)$$

with

$$e^{4\phi\eta} = \exp\left[4\alpha \frac{\mu c^2}{\hbar c} \left(\frac{Z_a Z_b}{k} \right) \arctan\left(\frac{k}{\beta} \right) \right], \qquad (28)$$

where $\widetilde{S}_0(E) = 4\pi^2\lambda/[R_B(\beta^2 + k^2)^2]$ with $R_B = \hbar^2/(2\mu Z_a Z_b e^2)$. The exponential factor $e^{4\phi\eta}$ (enhancement factor) can be applied to both light-nuclei reactions (small Z_a and Z_b) and heavy-ion reactions (large Z_a and Z_b) such as sub-barrier heavy-ion fusion reactions where $e^{4\phi\eta}$ can be very large. The use of a general form for $g(r) = e^{-\beta r}(\sum_{i=0}^N c_i r^i)$ instead of $e^{-\beta r}$ in Eq. (22) also leads to the same enhancement factor $e^{4\phi\eta}$. Therefore, the enhancement factor $e^{4\phi\eta}$ is independent of shape of the separable function $g(r)$ used in Eq. (22).

The enhancement factor $e^{4\phi\eta}$ is obtained together with the Gamow factor from our derivation and can be regarded as a modification of the Gamow factor, affecting it only at low energies, or as a part of the S factor if we still wish to keep the conventional formula, Eq. (1). We note that the

reaction cross section for a transition from elastic channel 1 to reaction channel 2 can be written in the form

$$\sigma_0(E) \propto |T_{12}(E)|^2, \qquad (29)$$

where

$$T_{12}(E) \propto \int g(r) \psi_0^c(r) dr, \qquad (30)$$

without using the optical theorem and the Feshbach procedure [23]. However, our previous derivation provides some qualitative justification that $g(r)$ can be real and a slowly varying function of energy for the nonresonant case, and it also provides a physical interpretation that $g(r)$ is related to the imaginary part of the elastic scattering amplitude.

IV. EXAMPLES OF AN APPLICATION

In this section, as examples of an application, we calculate the astrophysical S factors $S(E)$ for $^7\text{Li}(p,\alpha)^4\text{He}$, $^6\text{Li}(d,\alpha)^4\text{He}$, and $^6\text{Li}(p,\alpha)^3\text{He}$ reactions using the experimental data [14] and the analytic formula for $\sigma(E)$ based on the optical theorem formulation. Recent results for $\sigma(E)$ from laboratory beam experiments for nuclear reactions involving light nuclei at low energies (>3 keV) show that the extracted $S(E)$ increases toward lower energies instead of being a constant, indicating the possibility of a significant effect of the electron screening.

In the previous analyses [14] of $^7\text{Li}(p,\alpha)^4\text{He}$, $^6\text{Li}(d,\alpha)^4\text{He}$, and $^6\text{Li}(p,\alpha)^3\text{He}$ fusion reactions, $S(E)$ in Eq. (1) is modified to $S_G(E)$ by an enhancement factor $f_U(E) \approx \exp(\pi\eta U/E)$, which is attributed to the electron screening effect, and $S_G(E)$ is parametrized as

$$S_G(E) \approx f_U(E) S(E) \approx f_U(E) \sum_{i=0}^n c_i E^i, \qquad (31)$$

where U is the electron screening energy and $S(E) = \sum_{i=0}^n c_i E^i$ is the bare nuclear contribution. The fits of Eq. (31) (with up to three or four terms) to the experimental data [14] yield unphysical values of U, which are substantially larger than the expected values from the Thomas-Fermi model [26,27], given by

$$U = 30.7 Z_a Z_b (Z_a^{2/3} + Z_b^{2/3})^{1/2} \text{ eV}, \qquad (32)$$

which is obtained using the screening distance $a = 0.8853 a_0 (Z_a^{2/3} + Z_b^{2/3})^{-1/2}$ with the Bohr radius a_0.

In our analysis of these reactions, we use the electron screening energy U ($=161.6$ eV) calculated from Eq. (32) instead of treating it as a free fitting parameter. Based on our cross-section formula, Eq. (27) with Eq. (28), we parametrize the S factor using the formula

$$S(E) \approx f_U(E) \widetilde{S}_N(E) \approx f_U(E) \frac{a + bE}{(\beta^2 + k^2)^2} e^{4\phi\eta}, \qquad (33)$$

where $\widetilde{S}_N(E)$ is the bare nuclear contribution. To fit the experimental data [14] to extract the S factor, we have used only two or three parameters (a, b, and β) for the energy

TABLE I. Parameters used in the calculations of the S factor, Eq. (33), and normalization factor N for $^7\text{Li}(p,\alpha)^4\text{He}$, $^6\text{Li}(d,\alpha)^4\text{He}$, and $^6\text{Li}(p,\alpha)^3\text{He}$ reactions.

Reactions	a (keV b fm^{-4})	b ($10^{-3} \times$b fm^{-4})	β (fm^{-1})	N	χ^2
$^7\text{Li}(p,\alpha)^4\text{He}$	0.412	0.699	0.3589	0.960	0.96 (0.50)[a]
$^6\text{Li}(d,\alpha)^4\text{He}$	1.043		0.1837	0.835	0.45
$^6\text{Li}(p,\alpha)^3\text{He}$	2.039		0.2193	0.835	2.47

[a]The χ^2 value of 0.50 will result if the data point at $E=80.58$ keV is eliminated from the χ^2 calculation.

range of 10 keV$<E<$500 keV. The parameters used in our calculations are given in Table I.

For 100 keV$<E<$500 keV, there are 5, 9, and 9 data points available from the inverse incident kinematics cases of H($^7\text{Li},\alpha)^4\text{He}$, D($^6\text{Li},\alpha)^4\text{He}$, and H($^6\text{Li},\alpha)^3\text{He}$ reactions, respectively, while for $E<$100 keV, there are 22, 21, and 24 data points available from the original incident kinematics cases of $^7\text{Li}(p,\alpha)^4\text{He}$, $^6\text{Li}(d,\alpha)^4\text{He}$, and $^6\text{Li}(p,\alpha)^3\text{He}$ reactions. We used different normalizations N between the inverse and the original cases. Our values of N agree fairly well with those given in Ref. [14]. The calculated results for the astrophysical S factor are shown in Figs. 1–3. Each figure includes both the original and inverse kinematics of each reaction. As shown in the figures and Table I, we have obtained good fitting results with χ^2 values less than 0.5 for the $^6\text{Li}(d,\alpha)^4\text{He}$ reaction. The χ^2 value is about 1.0 per data point for the $^7\text{Li}(p,\alpha)^4\text{He}$ reaction, but it reduces to $\sim$0.5 if we eliminate one data point at $E=80.58$ keV in the fitting procedure. For the $^6\text{Li}(p,\alpha)^3\text{He}$ reaction, the χ^2 value is 2.47 per data point, indicating a poor fit to the experimental data [14]. For this reaction, a value of the electron screening energy much larger than the expected value given by the Thomas-Fermi model [25,26], Eq. (32), is required for obtaining better fits. Our extrapolated values of $\widetilde{S}_N(0) = (6.7\pm0.2)\times10^{-2}$ MeV b and (25 ± 2) MeV b for the bare nuclear contribution to the S factor, $\widetilde{S}_N(E)$, in Eq. (33), at zero energy, $E=0$, are substantially larger than the previous results [14] of 5.9×10^{-2} MeV b and 17.4 MeV b for the reactions $^7\text{Li}(p,\alpha)^4\text{He}$ and $^6\text{Li}(d,\alpha)^4\text{He}$, respectively. This

indicates large uncertainties for these S factors as inputs for calculations of the primordial nucleosynthesis and stellar evolution.

There are two main assumptions made for our calculations: (a) two open channels are assumed and (b) a nonresonant reaction is assumed. In the case of the $^6\text{Li}(d,\alpha)^4\text{He}$ reaction, there are several open channels ($^7\text{Li}+p$, $^7\text{Be}+n$, $^5\text{Li}+t$), requiring more than one term for $U_0(r,r')$, i.e., $U_0(r,r') = \Sigma_i \lambda_i g_i(r) g_i(r')$. Although the optical theorem gives us a total reaction cross section which includes the effect of all open channels, our use of a simple one-term approximation (corresponding to only two open channels) yields results which are consistent with the experimental data. The resonance state in ^8Be at an excitation energy of 22.2 MeV may be important if this state is assumed to consist of two closely lying mixed 2^+ states where the lower one has a large α width [28]. However, a good fit obtained from our calculation does not appear to support this hypothesis.

To understand the characteristics of the enhancement factor $e^{4\phi\eta}$ in the energy range considered, we plot this factor as a function of E for the $^6\text{Li}(d,\alpha)^4\text{He}$ reaction in Fig. 4 as an example. We can see from this figure that the enhancement factor $e^{4\phi\eta}$ has a value of $e^{2/(\bar{R}_B\beta)}=29.8$ at zero energy, decreases as E increases, and reaches to a value $e^{\pi/kR_B}=1$ for large E. Therefore, this factor can play an important role in explaining the anomalous enhancement of the S factor at low energies.

Electron screening effects have been investigated extensively by the Münster-Bochum group [14]. The determina-
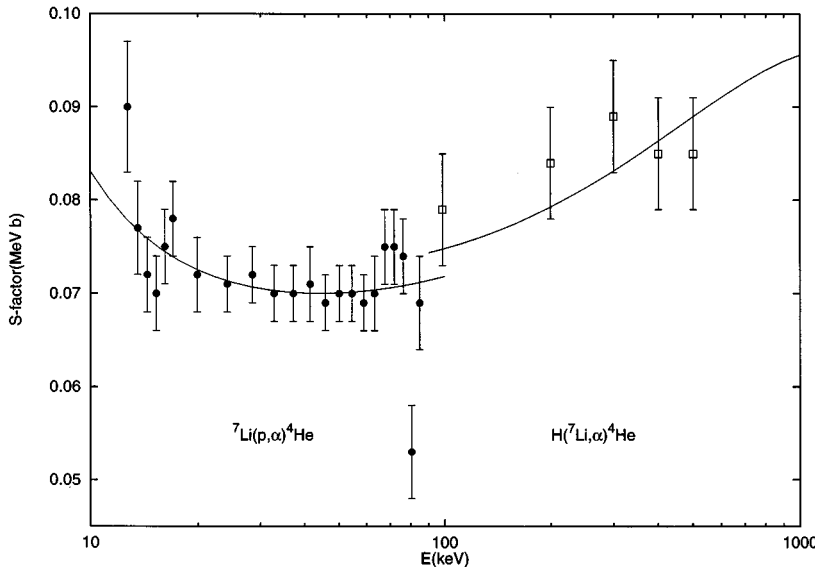

FIG. 1. Astrophysical S factor in the original incident kinematics (atomic target) for the $^7\text{Li}(p,\alpha)^4\text{He}$ reaction and in the inverse incident kinematics (molecular target) for the H($^7\text{Li},\alpha)^4\text{He}$ reaction. The experimental data are taken from Ref. [14]. The solid curves are our calculated S factor obtained from Eq. (33) using the parameter values given in Table I.
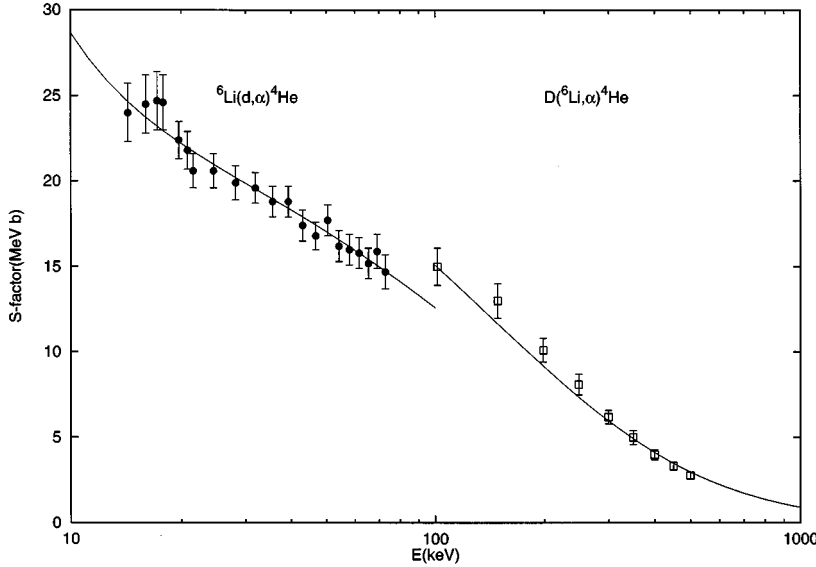
FIG. 2. Astrophysical S factor in the original incident kinematics (atomic target) for the $^6\text{Li}(d,\alpha)^4\text{He}$ reaction and in the inverse incident kinematics (molecular target) for the $\text{D}(^6\text{Li},\alpha)^4\text{He}$ reaction. The experimental data are taken from Ref. [14]. The solid curves are our calculated S factor obtained from Eq. (33) using the parameter values given in Table I.

tion of the electron screening energy requires not only high-precision measurements, but also an accurate determination of the effective energy in the target or, equivalently, of the energy loss. Recently, it has been claimed [29] that observed enhancements of the fusion $^3\text{He}(d,p)^4\text{He}$ cross section are due to an underestimate of the degraded beam energy arising from an overestimate of the stopping energy loss [30].

V. SUMMARY AND CONCLUSION

Based on the partial-wave optical theorem

$$\text{Im}\, f_l^{N(\text{el})} = \frac{k}{4\pi}\,(\sigma_l^{(r)} + \sigma_l^{N(\text{el})}) \approx \frac{k}{4\pi}\,\sigma_l^{(r)}$$

[Eqs. (5) and (6)], we have developed an optical theorem formulation of the nonresonant low-energy nuclear reactions between two charged nuclei. We have derived a relationship between the total cross section and the imaginary part of the elastic T matrix in the partial-wave form [Eq. (8)]. The use of a separable exponential form of the T matrix with several

parameters (strength, range, etc.) leads to an analytic formulation [Eqs. (26)–(28)] for the reaction cross section which exhibits explicitly the energy and charge dependence of the reaction cross section. In particular, it contains an enhancement factor [$e^{4\phi\eta}$, Eq. (28)] which increases toward low energies.

As examples of application, we have analyzed the $^7\text{Li}(p,\alpha)^4\text{He}$, $^6\text{Li}(d,\alpha)^4\text{He}$, and $^6\text{Li}(p,\alpha)^3\text{He}$ reactions [14] over the center-of-mass energy range of E from 10 to 500 keV using the parametrization of the cross section for a low-energy nuclear reaction based on the optical theorem formulation. Using the reasonable electron screening energy ($U = 161.6$ eV) obtained from the Thomas-Fermi model [26,27] instead of treating it as a free fitting parameter as done in the previous calculations, we obtain reasonably good fits to the astrophysical S factor extracted from the experimental data for $^7\text{Li}(p,\alpha)^4\text{He}$ and $^6\text{Li}(d,\alpha)^4\text{He}$ reactions. However, in the case of the $^6\text{Li}(p,\alpha)^3\text{He}$ reaction, our fits are poor in the lower-energy range (10 keV$<E<$100 keV), which implies that much larger values of the electron screen-
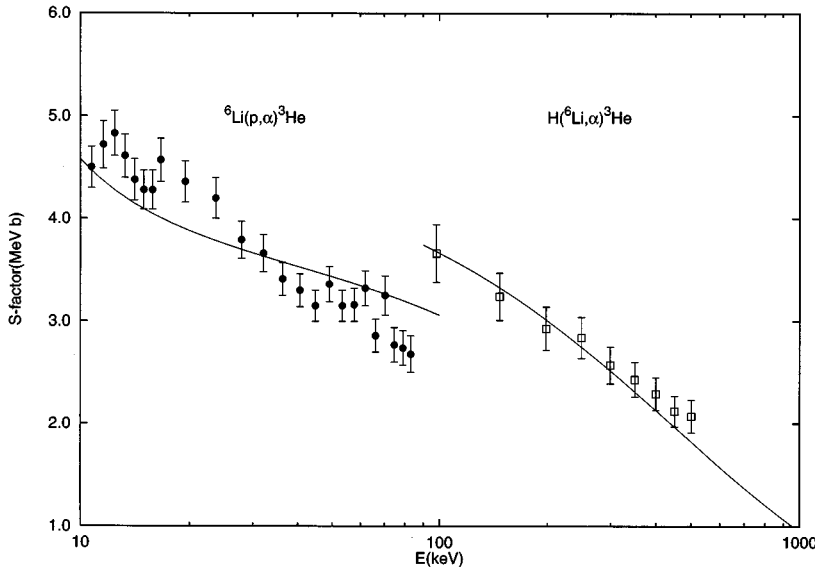

FIG. 3. Astrophysical S factor in the original incident kinematics (atomic target) for the $^6\text{Li}(p,\alpha)^3\text{He}$ reaction and in the inverse incident kinematics (molecular target) for the $\text{H}(^6\text{Li},\alpha)^3\text{He}$ reaction. The experimental data are taken from Ref. [14]. The solid curves are our calculated S factor obtained from Eq. (33) using the parameter values given in Table I.
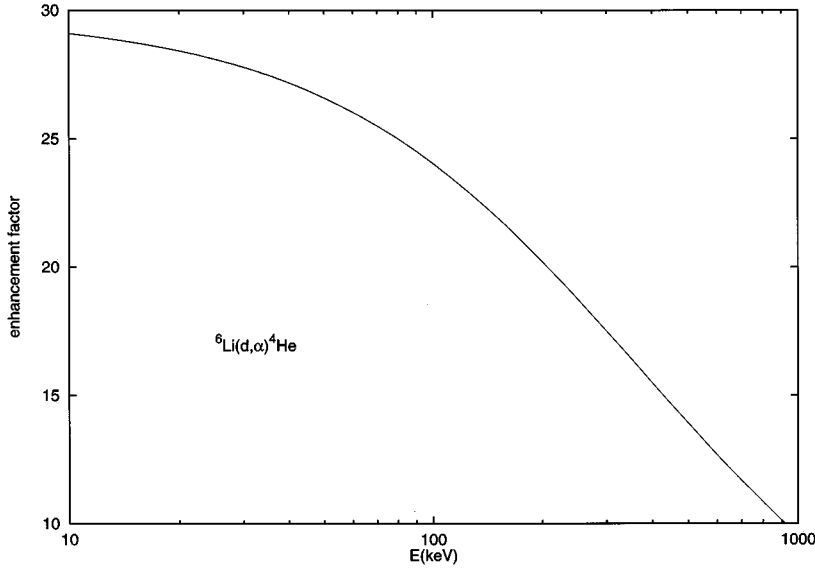
FIG. 4. Enhancement factor $e^{4\phi\eta}$, Eq. (28), as a function of energy E for the $^6\text{Li}(d,\alpha)^4\text{He}$ reaction.

ing energy than the estimated value obtained from Thomas-Fermi model [26,27] are needed to obtain a better fit to the experimental data.

Since the enhancement factor $e^{4\phi\eta}$ is a general feature arising from the strong and Coulomb interactions for bare nuclei without electrons (i.e., not from the electron screening effect) and since it is also applicable to all strong-interaction nonresonant nuclear reactions (excluding radiative capture reactions) between two charged nuclei occurring in the primordial nucleosynthesis and stellar evolutions, it may have a far-reaching astrophysical importance. Our investigation of the enhancement factor demonstrates that this factor may play an important role in understanding and/or explaining the experimental data which exhibit an anomalous increase of the S factor toward low energies.

APPENDIX A

To investigate the effective potential for a multichannel system, we consider a three-body problem with two open channels

$$1+(2,3)\rightarrow 1+(2,3),$$
$$1+(2,3)\rightarrow 2+(1,3). \tag{A1}$$

In terms of Faddeev-type components $\widetilde{\psi}_i$, the three-body Schrödinger equation $H\psi=E\psi$ can be rewritten as the Faddeev-type differential equation [25]

$$(E-H_0-V_1^C-V_{23})\widetilde{\psi}_1=(V_{12}+V_{23}-V_2^C)\widetilde{\psi}_2,$$
$$(E-H_0-V_2^C-V_{13})\widetilde{\psi}_2=(V_{12}+V_{13}-V_1^C)\widetilde{\psi}_1, \tag{A2}$$

where V_1^C and V_2^C are effective Coulomb potentials in elastic and fusion channels, respectively, and $\widetilde{\psi}_i$ is related to ψ by

$$\psi=\widetilde{\psi}_1+\widetilde{\psi}_2. \tag{A3}$$

Equation (A2) has a unique solution under some asymptotic boundary conditions if only binary channels are open [25]. We introduce the projection operators in terms of $\phi_1(\vec{r}_{23})$ and $\phi_2(\vec{r}_{13})$, which describe the bound states for subsystems (2,3) and (1,3), respectively:

$$P_1=|\phi_1\rangle\langle\phi_1|, \quad P_2=|\phi_2\rangle\langle\phi_2|,$$
$$Q_1=1-P_1, \quad Q_2=1-P_2. \tag{A4}$$

The use of operators P_i and Q_i, Eq. (A2), yields the following equation for the two-channel case:

$$(E-H_0-V_{23}-\hat{V}_{11})P_1|\widetilde{\psi}_1\rangle=\hat{V}_{12}P_2|\widetilde{\psi}_2\rangle,$$
$$(E-H_0-V_{13}-\hat{V}_{22})P_2|\widetilde{\psi}_2\rangle=\hat{V}_{21}P_1|\widetilde{\psi}_1\rangle, \tag{A5}$$

where the effective potential operators $\hat{V}_{ij}$ are given by

$$\hat{V}_{ij}=P_i\left\{a_{ij}+\sum_{k=1}^2 a_{ik}G_kA_{kj}\right\}P_j, \tag{A6}$$

$$G_k=Q_k\frac{1}{Q_k(E-H_0-V_k-A_{kk})Q_k}Q_k,$$

$$V_k\equiv V_{ij} \quad (i\neq j\neq k),$$

$$a_{ii}=V_i^C, \tag{A7}$$

$$a_{12}=V_{12}+V_{23}-V_2^C, \quad a_{21}=V_{12}+V_{13}-V_1^C,$$

and

$$A_{11}=a_{12}Q_2\frac{1}{Q_2(E-H_0-V_2-a_{22})Q_2}Q_2a_{21}+a_{11},$$

$$A_{12}=a_{12}Q_2\frac{1}{Q_2(E-H_0-V_2-a_{22})Q_2}Q_2a_{22}+a_{12},$$

$$A_{21}=a_{21}Q_1\frac{1}{Q_1(E-H_0-V_1-a_{11})Q_1}Q_1a_{11}+a_{21}, \tag{A8}$$

$$A_{22} = a_{21} Q_1 \frac{1}{Q_1(E - H_0 - V_1 - a_{11})Q_1} Q_1 a_{12} + a_{22}.$$

Equation (A5) can be arranged to obtain Eqs. (10) and (11). In the case of the RGM approximation, we have

$$\hat{V}_{ij}^{\mathrm{RGM}} = P_i a_{ij} P_j. \tag{A9}$$

Since a_{ij} given by Eq. (A8) are independent of energy, $\hat{V}_{ij}^{\mathrm{RGM}}$ in Eq. (A9) is also energy independent.

APPENDIX B

In this appendix, we show that the imaginary part of the T matrix is separable and symmetric. We can rewrite Eq. (20) as

$$\begin{aligned}
T(r,r',E) = & V_{\mathrm{Re}}^S(r,r',E) - i V_{\mathrm{Im}}^S(r,r',E) \\
& + \int [V_{\mathrm{Re}}^S(r,r'',E) - i V_{\mathrm{Im}}^S(r,r'',E)] \\
& \times G(r'',r''',E) T(r''',r',E) dr'' dr''',
\end{aligned} \tag{B1}$$

where

$$G(r,r',E) = -\frac{2\mu}{\hbar^2 k} \begin{cases} X(r)[Y(r') + iX(r')], & r < r', \\ [Y(r) + iX(r)]X(r'), & r' < r, \end{cases} \tag{B2}$$

with two linearly independent solutions $X(r)$ and $Y(r)$ in the absence of the strong potential $(V_{\mathrm{Re}}^S - i V_{\mathrm{Im}}^S)$ in Eq. (19).

Equation (B2) can be written in the form

$$G(r,r',E) = g(r,r',E) - i\frac{2\mu}{\hbar^2 k} X(r)X(r'), \tag{B3}$$

with $\mathrm{Im}\, g(r,r',E) = 0$.   Using   Eq.   (B3)   and $X(r) = C_0(\eta)(kr)e^{-ikr}M(1 - i\eta, 2, 2ikr)[C_0^2(\eta) = 2\pi\eta/(e^{2\pi\eta} - 1)$ and M is the Kummer's function], we obtain

$$\begin{aligned}
& -i\frac{2\mu}{\hbar^2 k} \int [V_{\mathrm{Re}}^S(r,r'',E) - i V_{\mathrm{Im}}^S(r,r'',E)] \\
& \times X(r'')X(r''')T(r''',r',E) dr'' dr''' \\
& \sim O(e^{-2\pi\eta}).
\end{aligned} \tag{B4}$$

Therefore, for the case of low energies $e^{-2\pi\eta} < 1$, we can use Eq. (B4) and rewrite Eq. (B1) as

$$\begin{aligned}
T(r,r',E) = & V_{\mathrm{Re}}^S(r,r',E) - i V_{\mathrm{Im}}^S(r,r',E) \\
& + \int [V_{\mathrm{Re}}^S(r,r'',E) - i V_{\mathrm{Im}}^S(r,r'',E)] \\
& \times g(r'',r''',E) T(r''',r',E) dr'' dr'''
\end{aligned} \tag{B5}$$

or

$$T(r,r',E) = T_{\mathrm{Re}}(r,r',E) - i T_{\mathrm{Im}}(r,r',E), \tag{B6}$$

where $T_{\mathrm{Im}}(r,r',E)$ satisfies

$$T_{\mathrm{Im}}(r,r',E) = \langle r|\widetilde{V}|r'\rangle + \int \langle r|\hat{K}|r''\rangle T_{\mathrm{Im}}(r'',r',E) dr'', \tag{B7}$$

with

$$\widetilde{V} = V_{\mathrm{Im}}^S[1 + \hat{g}(1 - V_{\mathrm{Re}}^S \hat{g})^{-1} V_{\mathrm{Re}}^S], \tag{B8}$$

$$\hat{K} = [V_{\mathrm{Re}}^S - V_{\mathrm{Im}}^S \hat{g}(1 - V_{\mathrm{Re}}^S \hat{g})^{-1} V_{\mathrm{Im}}^S]\hat{g}, \tag{B9}$$

$$\langle r|\hat{g}|r'\rangle \equiv g(r,r',E),$$

$$\langle r|V_{\mathrm{Im}}^S|r'\rangle \equiv V_{\mathrm{Im}}^S(r,r',E),$$

and

$$\langle r|V_{\mathrm{Re}}^S|r'\rangle \equiv V_{\mathrm{Re}}^S(r,r',E).$$

We can write the solution of Eq. (B7) as

$$T_{\mathrm{Im}}(r,r',E) = \int \Gamma(r,r'',E)\langle r''|\widetilde{V}|r'\rangle dr'', \tag{B10}$$

where Γ is the resolvent of the operator $\hat{K}$. Equation (B10) shows that T_{Im} is separable if $\widetilde{V}$ is separable. Since V_{Im}^S is separable, $\widetilde{V}$ is separable due to Eq. (B8), and hence T_{Im} is also separable due to Eq. (B10).

To prove that $T(r,r',E)$ is symmetric, we write T as

$$T(r,r',E) = T_S(r,r',E) + T_A(r,r',E), \tag{B11}$$

where

$$\begin{aligned}
T_S(r,r',E) &= T_S(r',r,E), \\
T_A(r,r',E) &= -T_A(r',r,E).
\end{aligned} \tag{B12}$$

We can rewrite Eq. (20) as

$$\begin{aligned}
T(r,r',E) &= V^S(r,r',E) + \int\int V^S(r,r'',E)G(r'',r''',E)T(r''',r',E) dr'' dr''' \\
&= V^S(r,r',E) + \int\int T(r,r'',E)G(r'',r''',E)V^S(r''',r',E) dr'' dr'''.
\end{aligned} \tag{B13}$$

Substituting Eqs. (B11) and (B12) into Eq. (B13), we obtain

$$T_S(r,r',E) \pm T_A(r,r',E) = V^S(r,r',E) + \int\int V^S(r,r'',E)G(r'',r''',E)[T_S(r''',r',E) \pm T_A(r''',r',E)]dr''dr'''$$

$$= V^S(r,r',E) + \int\int [T_S(r,r'',E) \pm T_A(r,r'',E)]G(r'',r''',E)V^S(r''',r',E)dr''dr''', \quad \text{(B14)}$$

which can be written as

$$T_S = V^S + V^S G T_S = V^S + T_S G V^S \quad \text{(B15)}$$

and

$$T_A = V^S G T_A = T_A G V^S. \quad \text{(B16)}$$

From Eq. (B16), we have $T_A = 0$, and hence $T(r,r',E)$ is symmetric.

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