Equivalent linear two-body method for Bose-Einstein condensates in time-dependent harmonic traps

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The recently developed time-independent effective linear two-body method [J. Phys. B: At., Mol. Opt. Phys. 33, 55 (2000)] has been generalized for time-dependent traps. The method is used to describe the dynamics of trapped Bose-Einstein condensates beyond the Thomas-Fermi regime. The calculated aspect ratios after ballistic expansion are found to be in good agreement with experimental data obtained recently by Görlitz et al. [Phys. Rev. Lett. 87, 130402 (2001)].

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I. INTRODUCTION

The newly created Bose-Einstein condensates (BEC) of weakly interacting alkali-metal atoms [1] stimulated a number of theoretical investigations (see recent review [2]). According to the Hohenberg theorem [3], the BEC is impossible in (one-dimensional) 1D and 2D homogeneous Bose gases. But BEC can occur in inhomogeneous systems, for example, in atomic traps [4]. The theoretical aspects of the BEC in highly elongated shaped traps (quasi-one-dimensional regime) have been reported in many papers [5–13]. The Gross-Pitaevskii (GP) equation [14] is widely used to describe the experimental results for BEC. In Ref. [5] it was found that the GP predictions for nonlinear dynamics (the aspect ratio after ballistic expansion) are in good agreement with those observed in a recent experiment [15]. We note that, a priori, it was not obvious that the GP equation gives the correct description of the nonlinear dynamics of the quasi-1D BEC.

Recently, an alternative method of equivalent linear two-body (ELTB) equations for many-body systems has been developed based on the variational principle [16–19]. It was shown that the ELTB method gives a good result for the ground state of Bose-condensed atoms in harmonic traps. The purpose of this work is to generalize the ELTB method [16–19] for the dynamics of trapped Bose-condensed gases. A recently developed approximation [5] provides the possibility of avoiding extensive numerical integration of the time-dependent ELTB equation. As an example of its application, this approximation is used to describe the ballistic expansion of the BEC after the cigar-shaped trap is switched off. The calculated aspect ratios are found to be in good agreement with the GP calculations and with the recent experimental results [15].

In Sec. II we derive the ELTB method for the time-independent trap. The accuracy of the ELTB method is confirmed by numerical computations. Section III considers the large-\( N \) limit. In Sec. IV, we developed an analytical formula for the lower bounds to the ELTB ground-state energy. Section V develops the analytical approximation for the time-dependent ELTB equation. We conclude the paper in Sec. VI with a brief summary.

II. TIME-INDEPENDENT TRAP

For the stationary \( N \)-body system, our method for obtaining the ELTB equation consists of following two steps.

The first step is to give the \( N \)-body wave function \( \psi(r_1,r_2,...) \) a particular functional form,

\[
\psi(r_1,r_2,...) = \tilde{\psi}(\zeta_1,\zeta_2,\zeta_3),
\]

where \( \zeta_1, \zeta_2, \) and \( \zeta_3 \) are known functions. We limit \( \zeta \)'s to three variables in order to obtain the ELTB equation, since a relative motion in the two-body problem depends on one vector described by three component variables. We note that approximation (1) allows us to study systems that are not spherically symmetric. The second step is to derive an equation for \( \tilde{\psi}(\zeta_1,\zeta_2,\zeta_3) \) by requiring that \( \tilde{\psi} \) must satisfy a variational principle

\[
\delta \langle \tilde{\psi} | H | \tilde{\psi} \rangle = 0
\]

with a subsidiary condition \( \langle \tilde{\psi} | \tilde{\psi} \rangle = 1 \). \( H \) is the Hamiltonian. This leads to a linear two-body equation from which both eigenvalues and eigenfunctions can be obtained.

To fix collective coordinates \( \zeta_1, \zeta_2, \) and \( \zeta_3 \), we note that the hyper radius

\[
\rho^2 = \sum_{i}^{N} (x_i^2 + y_i^2 + z_i^2)
\]

for an isotropic case [16] and also collective variables

\[
x^2 = \sum_{i}^{N} x_i^2, \quad y^2 = \sum_{i}^{N} y_i^2, \quad z^2 = \sum_{i}^{N} z_i^2,
\]

for an anisotropic case [17–19], yield good results for the dilute BEC of atoms in harmonic traps for both positive and negative scattering length. This success motivates us to introduce more general collective variables:

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\[
H = -\frac{\hbar^2}{2m} \sum_{i=1}^{N} \Delta_i + \frac{m}{2} \sum_{i=1}^{N} \left[ \omega_i^2 (x_i^2 + y_i^2) + \omega_i^2 z_i^2 \right] + \sum_{i<j} V_{\text{int}}(\vec{r}_i - \vec{r}_j).
\]

We use the Fermi pseudopotential approximation for \( V_{\text{int}} \),
\[
V_{\text{int}}(\vec{r}_i - \vec{r}_j) = \frac{4\pi \alpha^2}{m} \delta(\vec{r}_i - \vec{r}_j),
\]

where \( \alpha \) is the scattering length. For the eigenfunction \( \psi \) of \( H \), we assume the following form:
\[
\psi(\mathbf{r}_1, \ldots, \mathbf{r}_N) \approx \tilde{\psi}(r,z),
\]

where \( r^p = \sum_{i=1}^{N} (x_i^p + y_i^p) \) and \( z^q = \sum_{i=1}^{N} z_i^q \). The ELTB method leads to the equation for \( \tilde{\psi} \),
\[
\left[ H_0 + N^{1-2p} \alpha_p r^2 + N^{1-2q} \alpha_q z^2 + N^{2p+1+q} \gamma \frac{\partial^2}{r^2} \right] \tilde{\psi} = E \tilde{\psi},
\]

where
\[
H_0 = \frac{\hbar^2}{2m} \left( \frac{\partial^2}{\partial r^2} + \frac{2N-1}{r} \frac{\partial}{\partial r} \right) - \frac{\hbar^2}{2m} N^{1-2q}
\times \left( \frac{\partial^2}{\partial z^2} + \frac{N-1}{z} \frac{\partial}{\partial z} \right),
\]

\[
m_p = m \Gamma(1/p)(2/p)^{2-2p} \frac{\bar{\gamma}(2N/p,2-2p,0)}{2 \Gamma(2-1/p)},
\]

\[
m_q = m \Gamma(1/q)(1/q)^{2-2q} \frac{\bar{\gamma}(N/q,2-2q,0)}{2 \Gamma(2-1/q)},
\]

\[
\alpha_p = \frac{m \Gamma(3/p) \omega_p^2}{\Gamma(1/p)(2/p)^{2p} \bar{\gamma}(2N/p,2p,0)},
\]

\[
\alpha_q = \frac{m \Gamma(3/q) \omega_q^2}{2 \Gamma(1/q)(1/q)^{2q} \bar{\gamma}(N/q,2q,0)},
\]

and
\[
\gamma = \frac{\pi \hbar^2 (N-1) \bar{\gamma}(2N/p,0,2-2p) \bar{\gamma}(N/q,0,1-1q)}{4m(1/p)^{2-2p}(1/q)^{1-1q} \Gamma^2(1/p) \Gamma(1/q) 2^{1/p}}.
\]

with

FIG. 1. Ground-state energy per particles, \( E/N \), of \( ^{87}\text{Rb} \) atoms in a trap with \( \lambda = \sqrt{8} \), in units of \( \hbar \omega_r \), as a function of the number of particles in the trap. Solid circles, diamonds, dashed line, and solid line represent the results of theoretical calculations using the ELTB method, the \( p=2, q=2 \) approximation, the variational Monte Carlo [21], and the GP equation [20], respectively.

\[
\bar{\gamma}(z,a,b) = z^b-a \frac{\Gamma(z+a)}{\Gamma(z+b)}.
\]

Equation (9) simplifies if we introduce the new function \( u(r,z) \),
\[
\tilde{\psi}(r,z) = \frac{u(r,z)}{r^{2N-10}/z^{N(N-1)/2}}.
\]

In terms of \( u(r,z) \) Eq. (9) reads
\[
\left[ -\frac{\hbar^2}{2m} \frac{\partial^2}{r^2} - \frac{\hbar^2}{2m} \frac{\partial^2}{\partial z^2} + V_{\text{eff}}(r,z) \right] u(r,z) = Eu(r,z).
\]

The effective potential \( V_{\text{eff}}(r,z) \) is given by formula
\[
V_{\text{eff}}(r,z) = \frac{\hbar^2 (2N-1)(2N-3)}{8m_N 2^{2p} r^2} + \frac{\hbar^2 (N-1)(N-3)}{8m_N 2^{2q} z^2} + \alpha_p N^{1-2p} r^2 + \alpha_q N^{1-2q} z^2 + \frac{N^{2p+1+q}}{r^2 z}.
\]
solution of the GP equation [20] are about 2%, and the difference between our results and the calculations [21] are less than 1%.

Reference [22] proves that the GP mean-field theory describes correctly the energy and particle density of a dilute 3D Bose gas in a trap to the leading order in the small parameter $\tilde{p}a^3$ (where $\tilde{p}$ is the mean density and $a$ is the scattering length) when $N$ is large but $a$ is small with fixed $Na$.

We note also that for the case of lower dimensions $d < 3$, it is known that the quantum-mechanical two-body problem provides a unique possibility of checking the validity of various approximations made for the Schrödinger equation describing interacting via short-range potentials. For this case we seek the ELTB wave function in the form of $(\tilde{r})$, where $\rho \tilde{p} = \Sigma_{i=1}^{N} |x_i|^p$. Using Eq. (2) we obtain in the leading order of $N \rightarrow \infty$,

$$E = -c^2 \frac{N(N^2 - 1)}{24}.$$  

(21)

Choosing an optimal value of $p$, which minimizes the energy, leads to

$$\frac{E}{c^2 N^3} = -0.0412172.$$  

(22)

On the other hand, for large $N$, we have from Eq. (20)

$$\frac{E}{c^2 N^3} = - \frac{1}{24} = -0.0416667.$$  

(23)

The relative error for the binding energy between Eqs. (22) and (23) is about 1%. Therefore, we have demonstrated that the ELTB method is a very good approximation for the MY $N$-body problem for large $N$.

In Ref. [19], it was shown that in the case of large $N$ the ground-state wave function of $N$ bosons confined in a harmonic anisotropic trap can be written in a separable form as

$$\Psi(\bar{r}_1, \bar{r}_2, ..., \bar{r}_N) = \eta(x, y, z) \chi(\Omega),$$  

(24)

where $x^2 = \Sigma_{i=1}^{N} x_i^2$, $y^2 = \Sigma_{i=1}^{N} y_i^2$, $z^2 = \Sigma_{i=1}^{N} z_i^2$, and $\Omega$ is a set of $(3N - 3)$ angular variables.

Equation (24) may explain why the ELTB results are expected to be valid and are so close to the GP results for 3D dilute systems ($\tilde{p}a^3 \ll 1$). However, Eq. (24) is valid also for nondilute systems, while the GP mean-field theory is proven to be applicable for dilute systems. Therefore, we may expect that the ELTB approach will not be quantitatively equivalent to the solution of the GP equation for these cases. In our future work, we hope also to investigate the large gas parameter regimes [26].

Here we note that if scattering length $a$ is larger than the van der Waals length $r_v$ [27], there is a regime when the Bose system is dilute, but with respect to $r_0$, $\tilde{p}r_0^3 \ll 1$ [28]. For these systems the three-body contributions, given by the Efimov effect [29], can become the dominant term of the energy functional [28].

III. LARGE-$N$ LIMIT

To consider the large-$N$ limit, we rescale variables $r$ and $z$ in Eq. (15),

$$r = N^{1/p} \bar{r}, \quad z = N^{1/q} \bar{z},$$  

(25)

and rewrite Eq. (16) as

$$-\frac{\hbar^2}{2m} \frac{\partial^2}{\partial \bar{r}^2} + \frac{\hbar^2}{2m} \frac{\partial^2}{\partial \bar{z}^2} + V_{\text{eff}}(\bar{r}, \bar{z}) u(\bar{r}, \bar{z}) = E u(\bar{r}, \bar{z}).$$  

(26)

In the large-$N$ limit, $\bar{y}$ in Eqs. (11)–(13) is of the order of unity and the expression for $V_{\text{eff}}(\bar{r}, \bar{z})$ simplifies to

$$V_{\text{eff}}(\bar{r}, \bar{z}) = -\frac{\hbar^2}{2m \bar{r}^2} \bar{r}^2 \bar{z} + \frac{\hbar^2}{8m \bar{z}^2} \bar{z}^2 + m \omega_z^2 \bar{z}^2 + m \omega_z^2 \bar{z}^2 + \frac{\hbar^2 a N}{\bar{r}^2} \gamma' \bar{r}^3 + \frac{\hbar^2 a N}{\bar{z}^2} \gamma' \bar{z}^2,$$  

(27)

where

$$m'_1 = \frac{\Gamma(1/p)(2/p)^{2-2p}}{2(2-1/p)}, \quad m'_2 = \frac{\Gamma(1/q)(2/q)^{2-2q}}{2(2-1/q)},$$  

(28)

and

$$\alpha'_1 = \frac{\Gamma(3/p)}{\Gamma(1/p)(2/p)^{2p}}, \quad \alpha'_2 = \frac{\Gamma(3/q)}{2\Gamma(1/q)(1/q)^{2q}}.$$  

(29)
Quantum fluctuations are unimportant in the limit $N \to \infty$, and the most significant contribution to the ground-state energy is given by

$$E = NV_{\text{eff}}(r_0, z_0),$$  
(31)

where $r_0$ and $z_0$ are to be obtained from

$$\frac{\partial V_{\text{eff}}(r_0, z_0)}{\partial r_0} = \frac{\partial V_{\text{eff}}(r_0, z_0)}{\partial z_0} = 0.$$  
(32)

Obviously Eq. (31) fails if the effective potential does not possess a minimum.

Instead of variables $\bar{r}$ and $\bar{z}$, we introduce the new quantities

$$r_i = \bar{r}/a_\perp, \quad z_i = \bar{z}/a_\perp,$$  
(33)

where $a_\perp = \sqrt{\hbar/(m\omega_\perp)}$.

On making the substitution (33), Eqs. (27) and (30) become

$$V_{\text{eff}}(r_i, z_i) = \hbar \omega_\perp \left[ \frac{1}{2m_1 r_i^2} + \frac{1}{8m_2 z_i^2} + \alpha'_r r_i^2 + \lambda^2 \alpha'_z z_i^2 \right] + N(a/a_\perp) \frac{\gamma'}{2} \left[ \frac{1}{r_i z_i} \right],$$  
(34)

with

$$\alpha'_r r_i^2 - N(a/a_\perp) \frac{\gamma'}{r_i z_i} = \frac{1}{2m_1 r_i^2},$$  

$$2\lambda^2 \alpha'_z z_i^2 - N(a/a_\perp) \frac{\gamma'}{r_i z_i} = \frac{1}{4m_2 z_i^2},$$  
(35)

and $\lambda = \omega_z / \omega_\perp$.

In the case of large $N(a/a_\perp)$, solutions of Eqs. (35)

$$r_i = [N(a/a_\perp) \gamma'/ (2\lambda^2 \alpha'_z/\alpha'_r)]^{1/5},$$  

$$z_i = [N(a/a_\perp) \gamma' (\alpha'_z/2\lambda^2 \alpha'_r)]^{2/5} \left( \frac{1}{\alpha'_z} \right),$$  
(36)

give the ground-state energy

$$E = \frac{5}{2^{1/5}} (\alpha'_r \gamma' \alpha'_z)^{1/5}.$$  
(37)

Optimal values of $p=4$ and $q=4$ minimize the energy, Eq. (37), and we have

$$E = \frac{5}{2^{1/5}} (\alpha'_r \gamma' \alpha'_z)^{1/5} = 1.08199.$$  
(38)

For the case of large $N$, one can obtain an essentially exact expression for the ground-state energy by neglecting the kinetic-energy term in the Ginzburg-Pitaevskii-Gross equation (the Thomas-Fermi approximation) [30,31] as

$$E = \frac{5}{2^{1/5}} \left( \frac{15}{8} \right)^{1/5} = 1.05506.$$  
(39)

Comparing Eq. (38) with Eq. (39), we can see that for the case of large $N$ the ELTB method ($p=4, q=4$) is a very good approximation, with a relative error of about 2.5% for the binding energy (note that the $p=2, q=2$ case gives about 8.2% error for the binding energy).

**IV. LOWER BOUNDS**

In this section we formulate a lower-bound method for the solution of the ELTB equation (9). Following Ref. [5], we introduce auxiliary Hamiltonians

$$H_\perp = -\frac{\hbar^2}{2m_1 N^{1-2p}} \left[ \frac{\partial^2}{\partial r_i^2} + \frac{2N-1}{r_i} \frac{\partial}{\partial r_i} \right] + N^{1-2p} \alpha_\perp r_i^2,$$

$$H_z = -\frac{\hbar^2}{2m_2 N^{1-2q}} \left[ \frac{\partial^2}{\partial z_i^2} + \frac{N-1}{z_i} \frac{\partial}{\partial z_i} \right] + N^{1-2q} \alpha_z z_i^2,$$

and

$$H_\perp = \hbar N \sqrt{2\gamma_\perp} \gamma_\perp / m_1 + N^{1-2p} \alpha_\perp (1 - \gamma_\perp) r_i^2,$$

$$H_z = \hbar N \sqrt{2\gamma_z} \gamma_z / m_z + N^{1-2q} \alpha_z (1 - \gamma_z) z_i^2,$$

where $\gamma_\perp$ and $\gamma_z$ are parameters, restricted by $0 \leq \gamma_\perp < 1$ and $0 \leq \gamma_z < 1$, respectively. Using these auxiliary Hamiltonians we write the ELTB energy functional as

$$E = \langle \tilde{\psi} | (H_\perp + H_z - H_\perp - H_z) | \tilde{\psi} \rangle$$

$$+ \langle \tilde{\psi} | \left( \tilde{H}_\perp + \tilde{H}_z + N^{2p+1-q} + \frac{1}{2} \frac{\gamma'}{r_i z_i} \right) | \tilde{\psi} \rangle.$$  
(42)

Omission of $(H_\perp + H_z - H_\perp - H_z)$ yields a lower bound for the ground-state energy. Projecting $| \tilde{\psi} \rangle$ on the complete basis states $| n \rangle$, obtained from

$$h | n \rangle = \epsilon_n | n \rangle,$$

where

$$h = H_0 + N^{1-2p} \alpha_\perp \gamma_\perp r_i^2 + N^{1-2q} \alpha_z \gamma_z z_i^2,$$

we get

$$\langle \tilde{\psi} | h | \tilde{\psi} \rangle = \sum_n \epsilon_n \langle \tilde{\psi} | n \rangle \langle n | \tilde{\psi} \rangle \geq \epsilon_1$$

$$= N h \left( \sqrt{2\gamma_\perp} \gamma_\perp / m_1 + \frac{1}{2} \sqrt{2\gamma_z} \gamma_z / m_z \right).$$  
(43)

Therefore a set of optimal values of parameters $\gamma_\perp$ and $\gamma_z$, which maximizes our lower bound, will yield an optimal lower-bound value for the ground-state energy given by
TABLE I. Calculated results for the lower bound $E_{\text{low}}/N$, Eq. (36), and for the ground-state energy per particle, $E/N$, ELTB Eq. (9), in units of $\hbar \omega_c$, for the same case as in Fig 1. $\Delta$ is defined as $\Delta = (E - E_{\text{low}})/E$.

<table>
<thead>
<tr>
<th>$N$</th>
<th>$E_{\text{low}}/N$</th>
<th>$E/N$</th>
<th>$\Delta$</th>
</tr>
</thead>
<tbody>
<tr>
<td>100</td>
<td>2.66093</td>
<td>2.66286</td>
<td>7.2$\times$10$^{-4}$</td>
</tr>
<tr>
<td>200</td>
<td>2.86633</td>
<td>2.86797</td>
<td>5.7$\times$10$^{-4}$</td>
</tr>
<tr>
<td>500</td>
<td>3.34259</td>
<td>3.34378</td>
<td>3.6$\times$10$^{-4}$</td>
</tr>
<tr>
<td>1000</td>
<td>3.91311</td>
<td>3.91392</td>
<td>2.1$\times$10$^{-4}$</td>
</tr>
<tr>
<td>2000</td>
<td>4.69683</td>
<td>4.69742</td>
<td>1.3$\times$10$^{-4}$</td>
</tr>
<tr>
<td>5000</td>
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<td>6.25529</td>
<td>5.8$\times$10$^{-5}$</td>
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<td>7.94076</td>
<td>7.94101</td>
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</tr>
<tr>
<td>15000</td>
<td>9.17208</td>
<td>9.17227</td>
<td>2.1$\times$10$^{-5}$</td>
</tr>
<tr>
<td>20000</td>
<td>10.1877</td>
<td>10.1879</td>
<td>2.0$\times$10$^{-5}$</td>
</tr>
</tbody>
</table>

Using this approximation we calculate the energy per particle, $E/N$, for the same case as in Fig. 1. The calculated results are compared with those obtained from the numerical solutions of the ELTB equation in Table I. These comparisons show that the analytical approximation, Eq. (44), gives excellent results. The difference between $E/N$, Eq. (44) and numerical solutions of the ELTB equation is less than 0.07% for $100 \leq N \leq 5000$ and less than 0.006% for $N > 5000$.

V. TIME-DEPENDENT TRAP

In this section, we consider $N$ identical boson atoms confined in a time-dependent harmonic trap with the Hamiltonian

$$H(t) = -\frac{\hbar^2}{2m} \sum_{i=1}^{N} \Delta_i + \frac{m}{2} \sum_{i=1}^{N} \left[ \omega^2_i(t)(x_i^2 + y_i^2) + \omega^2_z(t)z_i^2 \right]$$

$$+ \frac{4}{m} \sum_{i<j} \delta(\vec{r}_i - \vec{r}_j).$$

To obtain the wave function, we apply the variational principle

$$\delta A = 0,$$

where the action integral $A$ is given by

$$A = \int_{t_0}^{t_1} \langle \Psi | \frac{i\hbar}{\partial t} - H(t) | \Psi \rangle dt,$$

and $\Psi(r,z,t)$ is the trial wave function.

This generalizes the time-independent ELTB equation [16–19] for time-dependent traps and leads to the equation

$$i\hbar \frac{\partial \Psi}{\partial t} = \left[ H_0 + N^{1/2} \alpha_i(t)r^2 + N^{1/2} \alpha_z(t)z^2 \right.$$
$$\left. + N^{2/2 + 1/2} \frac{1}{r^2} \frac{\gamma}{r^2} \right] \Psi,$$

where

$$\alpha_i(t) = \frac{m \Gamma(3/\beta)(t)}{(1/\beta)(2/\beta)^{2\beta} \pi^{2/\beta}} N^{2/\beta} N^{2/\beta}$$
$$\alpha_z(t) = \frac{2 \Gamma(1/\beta)(1/\beta)^{2/\beta}}{\pi^{2/\beta}} N^{2/\beta} N^{2/\beta}$$

with the initial condition $\Psi(r,z,0) = \tilde{\psi}(r,z)$, where $\tilde{\psi}(r,z)$ is a ground-state solution of the time-independent ELTB equation (9) with $\alpha_i(0) = \alpha_i$, $\alpha_z(0) = \alpha_z$.

We substitute the following Eq. (49) into Eq. (47) [5,32,33]:

$$\Psi(r,z,t) = \frac{\Phi(r,\lambda_i(t),z,\lambda_z(t),t)}{\lambda_i(t) \lambda_z(t)}$$
$$\times \exp[-i\beta(t) + i(f_i(t)r^2 + f_z(t)z^2)],$$

where

$$f_i(t) = -\frac{\lambda_i(t)m^{1/2}N^{1/2}}{2\hbar \lambda_i(t)},$$
$$f_z(t) = -\frac{\lambda_z(t)m^{1/2}N^{1/2}}{2\hbar \lambda_z(t)},$$

and $\beta, \lambda_i$, and $\lambda_z$ are solutions of the following equations:

$$\hbar \beta = \frac{E}{\lambda_i^{1/2} \lambda_z} + \hbar N \sqrt{2 \alpha_i(0) \gamma_i / m_i} \frac{1}{\lambda_i} - \frac{1}{\lambda_z} \lambda_i \lambda_z,$$
$$+ \frac{\hbar N}{2} \sqrt{2 \alpha_z(0) \gamma_z / m_z} \frac{1}{\lambda_z} - \frac{1}{\lambda_i} \lambda_i \lambda_z,$$
$$\beta(0) = 0,$$

$$\frac{m_i}{2} \lambda_i = - \alpha_i(0) \lambda_i + \frac{\alpha_i(0) \gamma_i}{\lambda_i} + \frac{\alpha_i(0)(1 - \gamma_i)}{\lambda_i \lambda_z},$$
$$\frac{m_z}{2} \lambda_z = - \alpha_z(0) \lambda_z + \frac{\alpha_z(0) \gamma_z}{\lambda_z} + \frac{\alpha_z(0)(1 - \gamma_z)}{\lambda_i \lambda_z},$$
$$\lambda_i(0) = 1, \lambda_i(0) = 0, \lambda_z(0) = 1, \lambda_z(0) = 0.$$
By neglecting \((H_{\perp} - \bar{H}_{\perp})\) and \((H_z - \bar{H}_z)\) in Eq. (53), we obtain a generalization of the approximation of Ref. [5] to the time-dependent ELTB equation

\[
\bar{\Psi}(r,z,t) = \frac{\bar{\Psi}(r/\lambda_{\perp}(t), z/\lambda_z(t))}{\lambda_{\perp}^{N_{\perp}}(t) \lambda_z^{N_z}(t)} \times \exp[-i\beta(t) + i(f_{\perp}(t)r^2 + f_z(t)z^2)],
\]

where all the dynamics is in the evolution of the scaling parameters \(\lambda_{\perp}(t)\) and \(\lambda_z(t)\), Eq. (52).

The aspect ratio of the cloud in the approximation, Eq. (54) is given by

\[
R(t) = \sqrt{\frac{\lambda_{\perp}^2(t)}{\lambda_z^2(t)}} = \frac{\lambda_{\perp}(t)}{\lambda_z(t)} R(0).
\]

As an example, we consider the application of the above results to the experimental data with \(^{23}\text{Na}\) atoms obtained in the Ioffe-Pritchard-type magnetic trap with radial and axial trapping frequencies of \(\omega_{\perp}/(2\pi) = 360\) Hz and \(\omega_z/(2\pi) = 3.5\) Hz [15], respectively. In our analysis, we use \(a = 2.75\) nm, \(t = 4\) ms, and \(a/\alpha_\perp = 2.488 \times 10^{-3}\), where \(a_\perp = \sqrt{\hbar/m\omega_{\perp}}\). As in Ref. [5], we consider a sudden and total opening of the trap at \(t = 0\). For this case, Eqs. (52) become

\[
d^2\lambda_{\perp} \over d\tau = b_{\perp} \left( \frac{\gamma_{\perp}}{\lambda_{\perp}} + \frac{1 - \gamma_{\perp}}{\lambda_{\perp} \lambda_z} \right),
\]

\[
d^2\lambda_z \over d\tau = b_z \left( \frac{\gamma_z}{\lambda_z} + \frac{1 - \gamma_z}{\lambda_z \lambda_{\perp}} \right) \varepsilon^2,
\]

where \(\tau = \omega_{\perp}(0)t\) and \(\varepsilon = \omega_z(0)/\omega_{\perp}(0) \ll 1\), and

\[
b_{\perp} = \frac{p^2 \Gamma(3/p) \Gamma(2 - 1/p)}{\Gamma^2(1/p) \gamma(2N/p,2/p,0) \gamma(2N/p,2 - 2/p,0)},
\]

\[
b_z = \frac{q^2 \Gamma(3/q) \Gamma(2 - 1/q)}{\Gamma^2(1/q) \gamma(N/q,2,0,q,0) \gamma(N/q,2 - 2/q,0)}.
\]

To zeroth order in \(\varepsilon^2\), we have \(\lambda_z = 1\) and \(\lambda_{\perp} = \sqrt{1 + b_{\perp} \tau}\). For the experimental conditions [15], the terms in \(\varepsilon^2\) are negligible. Our calculated results for \(R(t)\) are compared with those obtained from the solution of the GP equation [5], with the Thomas-Fermi (TF) approximation, and with experimental data [15] in Fig. 2. This comparison shows that the present results, Eqs. (56), are in good agreement with the GP calculations and with the recent experimental results [15].

VI. SUMMARY AND CONCLUSION

In summary, we have generalized the time-independent ELTB method [16–19] to the time-dependent case. As examples of application, we have studied the problem of the ballistic expansion of the condensate after the cigar-shaped traps are switched off. The approximation developed in Ref. [5] provides a possibility of avoiding extensive numerical integrations of the time-dependent ELTB equation.

The calculated aspect ratios after ballistic expansion are found to be in a good agreement with experimental data obtained recently by a group at MIT.

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