Time-dependent density-functional theory for trapped strongly interacting fermionic atoms

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The dynamics of strongly interacting trapped dilute two-component Fermi gases (dilute in the sense that the range of interatomic potential is small compared with interparticle spacing) is investigated in a single-equation approach to the time-dependent density-functional theory. For the ground-state energy per particle of the system in the homogeneous phase, we have constructed an Padé parametrization based on Monte Carlo data and asymptotic behavior. Our numerical results for collective frequencies in the BCS-BEC crossover regime are in good agreement with recent experimental data obtained by the Duke University group. In addition, we show that the calculated corrections to the hydrodynamic approximation may be important, even for systems with a rather large number of atoms.

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The recently reported ultracold trapped Fermi gases with tunable atomic scattering length [1–11] in the vicinity of a Feshbach resonance stimulated a large number of theoretical investigations. Some of these works are based on the assumption that the properties of strongly interacting dilute Fermi gas at zero temperature are well described by the hydrodynamic approximation (HA) [12–15]

\[
\frac{\partial n}{\partial t} + \nabla \cdot (nv) = 0,
\]

\[
\frac{\partial \bar{v}}{\partial t} + \frac{1}{m} \nabla \left( V_{\text{ext}} + \frac{\partial [n\epsilon(n)]}{\partial n} + \frac{1}{2}mv^2 \right) = 0,
\]

where \( n \) is the density, \( \epsilon(n) \) is the ground-state energy per particle of the homogeneous system and \( \bar{v} \) is the velocity field.

In this paper the dynamics of strongly interacting trapped dilute Fermi gases (dilute in the sense that the range of interatomic potential is small compared with interparticle spacing) is investigated in the single equation approach to the time-dependent density-functional theory covering the whole crossover region at zero temperature. It is shown for the case of elongated cigar-shaped harmonic traps that the calculated corrections to the HA may be important even for systems with a rather large number of atoms.

We mention here Ref. [16] where an extension of the density-functional theory (DFT) to superconducting systems [17] was generalized to a number of nuclear and atomic systems. Let us consider a Fermi gas consisting of a 50–50 mixture of two different states confined in a harmonic trap \( V_{\text{ext}}(r) = (m/2)[\omega_x^2(x^2+y^2) + \omega_z^2z^2] \). In Eq. (2), the kinetic-energy density \( t(n) \) is approximated by the Thomas-Fermi (TF) kinetic-energy density \( t_{\text{TF}}(n) = (3/10)n\hbar^2k_F^2/m \), where \( k_F = (3\pi^2n)^{1/3} \). For slowly varying densities characterized by the condition \( |\nabla n|/n^{4/3} \ll 1 \), the kinetic energy density is well represented by the Kirzhnitz gradient expansion (KGE) [18] \( t(n) = t_{\text{TF}}(n) + t_{\phi}(n)/9 + \cdots \), where \( t_{\phi}(n) = (\hbar^2/8m)|\nabla n|^2/n \) is the original von Weizsäcker density (OWD) [19], which gives the entire kinetic energy density of noninteracting bosons.

In the case of large but finite number of atoms \( N \), the density \( n \) is not constant. At small distances the ratio \( |\nabla n|/n^{4/3} \) is small and both the Kirzhnitz correction and the OWD are negligible. On the contrary, near the surface the Hartree-Fock (HF) type densities are proportional to the square of the last occupied state. Therefore, the OWD is important in this case and it is expected to determine the asymptotic behavior of the density at large distances. It is also expected that the OWD is important in the case of the tight radial trapping, \( \lambda \ll 1 \). In Ref. [20], the OWD was considered as a correction to the TF kinetic-energy density.

Adding the OWD to \( t_{\text{TF}}(n) \) we have

\[
\frac{\partial \bar{v}}{\partial t} + \frac{1}{m} \nabla \left( V_{\text{ext}} + \frac{\partial [n\epsilon(n)]}{\partial n} + \frac{1}{2}mv^2 - \frac{\hbar^2}{2m} \frac{1}{n} \nabla^2 n \right) = 0.
\]

We define the density of the system as \( n(r,t) = |\Psi(r,t)|^2 \), and the velocity field \( \bar{v} \) as \( \bar{v}(r,t) = \hbar (\Psi^* \nabla \Psi - \Psi \nabla \Psi^*)/[2imn(r,t)] \). From Eqs. (1) and (3), we obtain the following nonlinear Schrödinger equation:

\[
i\hbar \frac{\partial \Psi}{\partial t} = -\frac{\hbar^2}{2m} \nabla^2 \Psi + V_{\text{ext}} \Psi + \frac{\partial [n\epsilon(n)]}{\partial n} \Psi,
\]

which is equivalent, to a certain extent, to the single equation approach of Deb et al. [21] to the time-dependent density-functional theory (TDDFT).

If the trap potential \( V_{\text{ext}} \) is independent of time, one can write \( \Psi(r,t) = \Phi(r) \exp(-\mu t/\hbar) \), where \( \mu \) is the chemical potential, and \( \Phi \) is normalized to the total number of particles, \( \int d^3r |\Phi|^2 = N \). Then Eq. (4) becomes
where the solution of the equation (5) minimizes the energy functional \( E = N(\Phi) - \frac{\hbar^2}{2m} \nabla^2 + V_{\text{ext}} + e(n)|\Phi|^2 \), and the chemical potential \( \mu \) is given by \( \mu = \partial E / \partial N \).

In order to take into account atoms lost by inelastic collisions, we model the loss by the rate equation

\[
\frac{dN}{dt} = -\int \chi(\vec{r}, t) d\vec{r},
\]

where \( \chi(\vec{r}, t) = \sum_{i=1}^{N_l} k_i n_i g_i(n) \), \( n_i g_i \) is the local \( l \)-particle correlation function and \( k_i \) is the rate constant for the \( l \)-body atoms loss. The generalization of Eq. (4) for the case of inelastic collisions reads [22–25]

\[
\frac{i\hbar}{\partial t} \Psi = -\frac{\hbar^2}{2m} \nabla^2 \Psi + V_{\text{ext}} + e(n)|\Psi|^2 - \frac{\hbar}{2} \sum_{i=1}^{N_l} k_i n_i^{-1} g_i(n) \Psi.
\]

For the negative \( S \)-wave scattering length between the two fermionic species \( a < 0 \) in the low-density regime \( k_F |a| \ll 1 \), the ground state energy per particle \( e(n) \) is well represented by an expansion in power of \( k_F |a| \) \[26\]

\[
e(n) = 2E_F \left[ \frac{3}{10} - \frac{1}{3\pi} k_F |a| + 0.055661(k_F |a|)^2 - 0.00914(k_F |a|)^3 + \cdots \right],
\]

where \( E_F = \hbar^2 k_F^2 / (2m) \). In the opposite regime, \( a \rightarrow -\infty \) (the Bertsch many-body problem, quoted in Ref. [27]), \( e(n) \) is proportional to that of the noninteracting Fermi gas

\[
e(n) = (1 + \beta) \frac{3}{10} \frac{\hbar^2 k_F^2}{m},
\]

where a universal parameter \( \beta \) \[10\] is estimated to be \( \beta = -0.56 \) \[28\].

In the \( a \rightarrow +0 \) limit the system reduces to the dilute Bose gas of dimers \[29\]

\[
e(n) = E_F \left[ -1/(k_F a) + a_{\omega} k_F (6\pi) + \cdots \right],
\]

where \( a_{\omega} \) is the boson-boson scattering length. While the BCS mean-field theories \[30\] predict \( a_{\omega} = 2a \) \[31\], a solution of four-fermion problem for contact scattering provided the value \( a_{\omega} = 0.6a \) \[32\].

Very little is known about the correct form of \( e(n) \) in the intermediate range. Therefore, a simple interpolation of the form \( e(n) = E_F P(k_F a) \) with a smooth function \( P(x) \) mediating between the known limits suggests itself as a pragmatic alternative.

In Ref. [33] it has been proposed a \([2/2]\) Padé approximant for the function \( P(x) \) for the negative \( a \)

\[
P(x) = \frac{E_{\text{mol}}}{2E_F} + \frac{\alpha_1 x + \alpha_2 x^2}{1 + \alpha_3 x + \alpha_4 x^2},
\]

where parameters \( \alpha \) are fixed by two continuity conditions at large \( x, 1/x \rightarrow 0 \), and by two continuity conditions at small \( x \). For example, \( \alpha_1 = 0.0316621, \alpha_2 = 0.0111816, \alpha_3 = 0.200149, \) and \( \alpha_4 = 0.0423545 \) for \( a_{\omega} = 0.6a \).

Figures 1 and 2 show the comparison between \([2/2]\) Padé approximations, Eqs. (10) and (11), and the lowest order constrained variational (LOCV) approximation \[34\] and the BCS mean-field theory for \( e(n) \). The LOCV calculations agree very well with the \([2/2]\) Padé approximation results on the BCS side \( (\alpha < 0) \). It is evident the difference between our results and the BCS mean-field theory calculations. For example, the BCS mean-field gives \( \beta = -0.41 \). We mention here that \( e(n)/E_F \) on the BCS side \( (\alpha < 0) \) and \( [E(n) + |E_{\text{mol}}|/2]/E_F \) on the BEC side \( (\alpha > 0) \) show a smooth monotonic behavior as a function of \( k_F a \).

The predictions of Eq. (5) with \( e(n) \) from Eq. (10) for the axial cloud size of strongly interacting \(^6\)Li atoms are shown in Fig. 3 \[35,36\]. It indicates that the TF approximation of the kinetic energy density is a very good approximation for the experimental conditions of Ref. [11], \( 4N \approx 10^5 \) (inclusion of the OWD gives a negligible effect, <0.5%) \[37–41\].
FIG. 2. The ground-state energy per particle $\epsilon(n)+|E_{\text{mol}}|/2$, in units of $\hbar^2k_F^2/(2m)\,$, as a function of the gas parameter $(k_Fa)^{-1}$. The dashed line, the dotted-dashed line and the solid line represent the results calculated using the BCS mean-field theory, the [2/2] Padé approximation (11) with $a_m=2a$ and $a_m=0.6a$, respectively.

It can be proved [24] that every solution of Eq. (4) is a stationary point of an action corresponding to the Lagrangian density

$$L_0 = \frac{i\hbar}{2} \left( \Psi^* \frac{\partial \Psi}{\partial t} - \Psi \frac{\partial \Psi^*}{\partial t} \right) + \frac{\hbar^2}{2m} \left( \nabla \Psi \right)^2 + \epsilon(n)n + V_{\text{ext}}n,$$

which for $\Psi=e^{i(k_Fx)n^{1/2}}(\tilde{r},t)$ can be rewritten as

$$L_0 = \hbar \phi n + \frac{\hbar^2}{2m} \left( \nabla \phi \right)^2 + \frac{\hbar^2}{2m} \left( \nabla \phi \right)^2 + \epsilon(n)n + V_{\text{ext}}n.$$

For a time-dependent harmonic trap, $V_{\text{ext}}(\tilde{r},t) = (m/2)\sum_{i=1}^{3}\omega_i^2(t)x_i^2$, a suitable trial function can be taken as

$$\Phi(\tilde{r},t) = \chi(t) = m/(2\hbar)\sum_{i=1}^{3}\phi_i(t)\eta_i(t),$$

TABLE I. Radial breathing mode frequencies $\nu=\omega_{\text{rad}}/(2\pi)$ of highly degenerate gas of $^6\text{Li}$ atoms near a $822\,\text{G}$ Feshbach resonance [36]. $B$ is applied magnetic field, $v_{\text{exp}}$ indicate experimental data from the Duke University group [1]. $\nu$ and $\nu_{\text{TF}}$ represent theoretical calculations that use Eq. (4) and the hydrodynamic approximation [Eqs. (1) and (2)], respectively. The trap parameters are $\omega_r=2\pi\times1549, \omega_{\theta}=2\pi\times70, \omega_z=2\pi\times70$.

<table>
<thead>
<tr>
<th>$B(G)$</th>
<th>$N$ (units of $10^5$)</th>
<th>$v_{\text{exp}}$ (Hz) [1]</th>
<th>$\nu$ (Hz)</th>
<th>$\nu_{\text{TF}}$ (Hz)</th>
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<tr>
<td>860</td>
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<td>910</td>
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<td>2792</td>
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</table>

Expanding Eq. (12) around equilibrium ($\phi_0=1$) we get the following equations for the collective frequencies $\omega$

$$\left( 2 + \frac{\kappa_i}{\omega_i^2} \right) \chi_i + \left( 1 + \frac{1}{2} \kappa_i + \chi_i \right) (y_1+y_2+y_3) = 0,$$

where $\kappa_i=4\langle T_i \rangle/(m\omega_i^2\langle x_i^2 \rangle)$ and $\chi_i=\int n_0^3 d^3r \phi_i^2/\langle m\omega_i^2\rangle$. In Table I we give the calculated values of the radial breathing mode frequency, $\nu=\omega_{\text{rad}}/(2\pi)$ of highly degenerate gas of $^6\text{Li}$ atoms near a Feshbach resonance at 822 G [41]. It can be seen from Table I, that the difference between two approximations, Eqs. (1), (2), and (4), is less than 0.75%, and both approximations give a very good agreement with experimental data of Ref. [1]. The parameter $\lambda N$ for this case is very large $\lambda N \gg 10^4$.

In Fig. 4, we present the calculations for the frequency of the radial compression mode $\omega_{\text{rad}}$ as a function of the dimensional parameter $(N/m\hbar^2\omega_0)^{-1}$ in the case of an anisotropic trap ($\omega_\theta=\omega_z=\omega_\lambda, \omega_r=\lambda$). One can easily see that the corrections to the hydrodynamic approximation (HA), Eqs. (1) and (2), are important even for relatively large $N$ and $\lambda N$.  

FIG. 3. Axial cloud size of strongly interacting $^6\text{Li}$ atoms after normalization to a non-interacting Fermi gas with $N=4\times10^5$ atoms as a function of the magnetic field $B$ [32]. The trap parameters are $\omega_r=2\pi\times640\,\text{Hz}, \omega_\theta=2\pi\times(600B/2\hbar G+32)^{1/2}\,\text{Hz}$. The solid line and dashed line represent the results of theoretical calculation that includes the OWD or uses the TF approximation for the kinetic energy density, respectively. The circular dots indicate experimental data from the Innsbruck group [11].
compared with experimental data.

The TF approximation reproduces the energy within accuracies less than 11 and 25% for $\lambda = 10^{-2}$ and $\lambda = 10^{-2}$, $N = 10^3$, respectively. The deviation from this behavior does not demonstrate the crossover to the 1D behavior, since $\lambda N > 1$ [42]. It demonstrates that the validity of the HA depends on the properties of the trap. In Ref. [43] it was shown that, for the case of isotropic trap, $\lambda = 1$, with $N = 20$ and $N = 240$, the TF approximation reproduces the energy within accuracies of 2 and 1%, respectively.

In Fig. 5, the calculated radial compressional frequency is compared with experimental data [1] in the BCS-BEC crossover region. There is a very good agreement between calculations and experimental data [1]. However our calculations for $\omega_{\text{rad}}$ disagree with experimental data of Ref. [44].

In the present paper, we have used Eq. (4). The next step is to develop the Kohn-Sham time-dependent DFT [45] for two-component Fermi gases in elongated traps ($\lambda \ll 1$), which we will consider in our future work.

In conclusion, we have investigated a possible improvement of the hydrodynamics approach by adding the quantum-pressure term to the TF kinetic energy, (4). We have also introduced Padé approximations for the equations of state of the uniform dilute Fermi gas across the BCS-BEC crossover, and have compared the predicted collective frequencies to experimental data. It is shown that the calculated corrections to the hydrodynamic approximation may be important even for systems with a rather large number of atoms.

Note added. Recently some papers [46] appeared in which the authors calculate the equation of state $\epsilon(n)$ using the quantum Monte Carlo method. Their results are in a good agreement with our Padé [2/2] approximation for both negative and positive scattering lengths.

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[22] The only difference from equations holding for bosons [23,24] is given by density dependence of $\epsilon(n)$. We do not consider three-body recombinations, since these processes play an important role near $p$-wave two-body Feshbach resonance [25].


[35] To calculate the ground-state density we have used a highly accurate variational approach of Ref. [36]. This method gives, for example, in unitary limit for the case of very large $N$ the value of energy $E/[N^{10/3},(1+\beta)^{1/3}]=1.08486$, which is very close to the exact value $3^{4/3}/4=1.08169$ (relative error is less than 0.3%).


[37] We have used the data from Ref. [38] to convert $a$ to $B$. We note here that in general a Feshbach resonance may lead to the density dependence of the effective interaction (for bosons cases see, for example, Refs. [39,40]). In Ref. [41], the resonant position was accurately determined to be $822\pm 3\, g$.


[40] V. A. Yurovsky, e-print cond-mat/0308465.


