The Generalized Dispersion Tensor Revisited: Theory and Calculation for Homogeneous and Heterogenous Porous Media

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1 Introduction

Over the last two decades a substantial number of researchers have studied the causes of anomalous dispersion. In the theory of porous media, it is well known that the concept of homogeneity is scale dependent; even a medium consisting of relatively uniform shaped and sized particles is not appropriately modeled by equations derived from the standardly proposed laws for homogeneous media (i.e. advection diffusion equation) until a sufficient portion of the media has been sampled. Cushman and Moroni [1] developed an equation governing the displacement of conserved particles under very general conditions using what has been termed the generalized hydrodynamic approach, which is based on molecular hydrodynamics [2]. The convolution flux presented in [1] is often overlooked in the literature, perhaps due to its generality. It is a theory providing a scale-dependent dispersion tensor, and in its most general form, is non-local in both space and time. Classical theories are at a fixed scale where required assumptions must be met. Due to the freedom from assumptions of scale, the result in [1] can be seen as a universal equation for dispersive processes.

2 Theoretical background

The transition density function for displacement, G, when appropriately normalized, can be viewed as the concentration from a point source. If one assumes all particles are identical then G is simply the expected value of a delta distribution:

$$G(\mathbf{x},t) = \left\langle \delta(\mathbf{x} - (\mathbf{X}(t) - \mathbf{X}(0))) \right\rangle \tag{1}$$

where $\mathbf{X}(t)$ is the particle position at time t and $\langle\ \rangle$ is the average with respect to a probability density on phase space. If this density is properly chosen, then G corresponds to the probability of a given particle displacement (or concentration for a point source) at a given time. The assumptions made for the following results are that the phase space system is incompressible, Hamiltonian, and at local equilibrium. Local equilibrium physically means that if an observer travels along the nonequilibrium mean of the trajectory, $\langle \mathbf{X}(t) \rangle$, then the world viewed from this position appears in equilibrium. At equilibrium the phase space probability density is independent of time. More details on the assumptions and derivations can be found in [1].

At local equilibrium, the flux governing G is nonlocal in space and time and takes the form

$$\mathbf{q} = -< \mathbf{v}(t) > G(\mathbf{x},t) + \int_0^t \int_{R^3} \mathbf{D}(\mathbf{y},t,\tau) \cdot \nabla_{\mathbf{x}-\mathbf{y}} G(\mathbf{x}-\mathbf{y}, t-\tau) d\tau d\mathbf{y} \tag{2}$$

where $\mathbf{D}(y,t,\tau)$ is the generalized dispersion tensor. This dispersion tensor represents all observed spreading including that caused by diffusion, hydrodynamic dispersion and any other factors not accounted for by mean convection. Note this equation contains a spatial convolution term; it is not a convolution in time, but is non-local.

For the purpose of testing the theory, a laboratory experiment was designed to allow the tracking of particles within a medium from a Lagrangian perspective. Porous media were constructed using Pyrex beads and cylinders in a vertical test section; a medium was classified as 'homogenous' if it was constructed of pieces of the same size and shape and otherwise classified as 'heterogeneous'. A mean flow in the vertical direction was induced by pumping glycerol into and out of the laboratory-scale medium. The laboratory set-up and basic statistics from the experiments are fully described in [3, 4], so here we focus only on the results and their interpretation.

References

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