Introduction to Percolation
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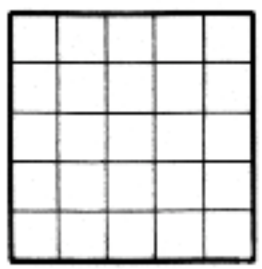
- What is percolation?
- The percolation threshold - connection with phase transitions and critical phenomena
- Fractals and fractal scaling
  - ➢ upscaling from small to large scales
- Properties
  - ➢ conductivity
  - ➢ fluid flow
  - ➢ strength
- Open issues

[Recommended reference: Introduction to Percolation Theory, by Stauffer and Aharoni]

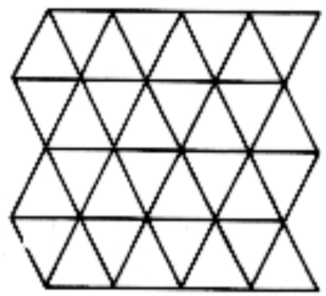
What is Percolation?

- Consider percolation on a lattice

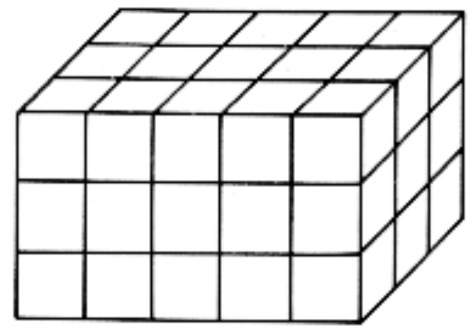
square (2D)



honeycomb (2D)



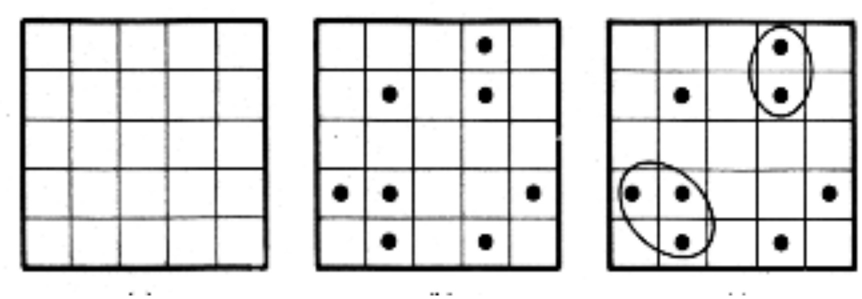
cubic (3D)



- Behavior depends on dimensionality (a lot) and lattice type (a little)
- Can also consider <u>continuum</u> percolation (more realistic for us, but not covered in these lectures)

What is Percolation?

- Start with an empty lattice - then occupy sites at random



- Connected occupied sites form clusters
- Percolation is about the properties of these clusters -- size, connectivity, etc.

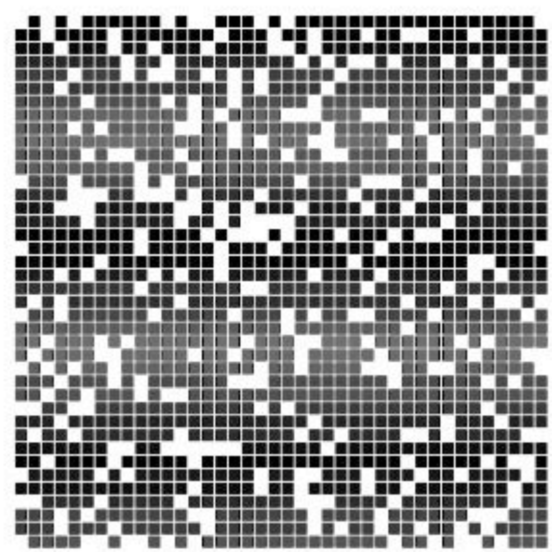
Consider connectivity across the lattice

- Connectivity depends on concentration of occupied sites = p
- Connectivity changes a p_c ($\approx$0.59 for site percolation on a square lattice)

p = 0.40

p = 0.60

p = 0.80



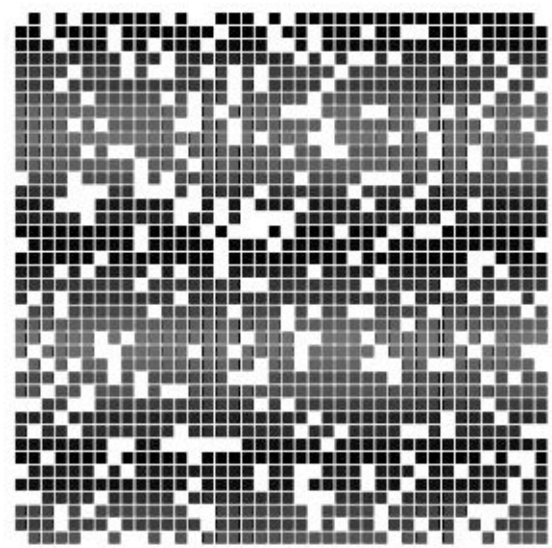
p_c is the "critical" concentration for percolation

- A "connectivity" phase transition occurs at $p_c \sim 0.59$
- A spanning cluster first appears at p_c
- Many properties are singular at p_c

$p = 0.40$

$p = 0.60$

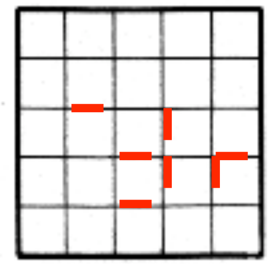
$p = 0.80$

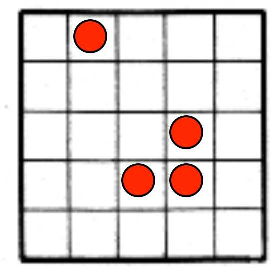


p_c depends on lattice type

- p_c is also different for site versus bond percolation

Lattice	Site	Bond
Honeycomb	0·6962	0·65271
Square	0·592746	0·50000
Triangular	0·500000	0·34729
Diamond	0·43	0·388
Simple cubic	0·3116	0·2488
BCC	0·246	0·1803
FCC	0·198	0·119
$d=4$ hypercubic	0·197	0·1601
$d=5$ hypercubic	0·141	0·1182
$d=6$ hypercubic	0·107	0·0942
$d=7$ hypercubic	0·089	0·0787





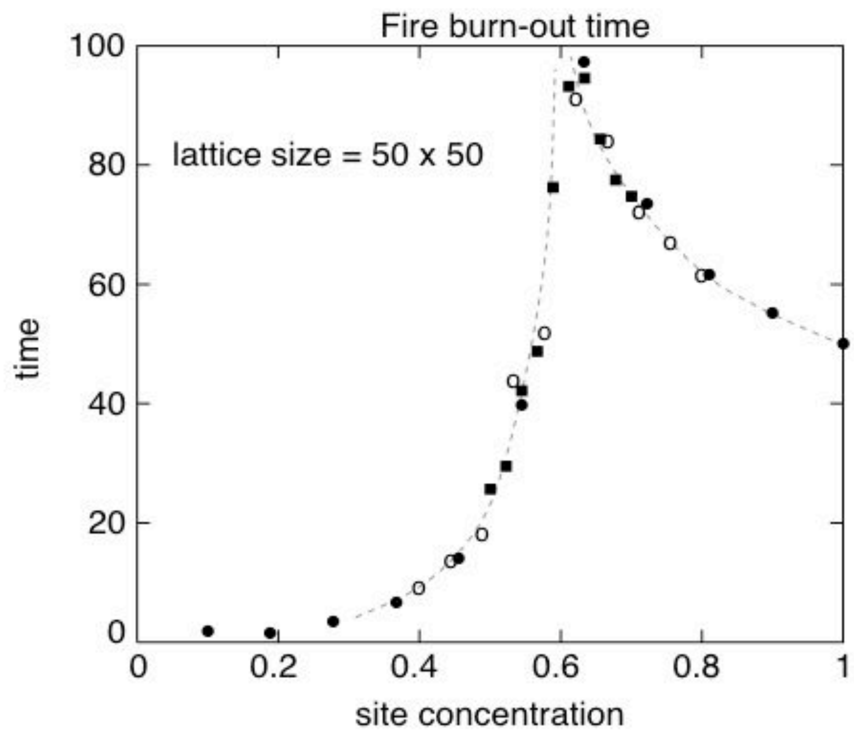
Why is p_c special?

- Consider the forest fire problem
- Each occupied site is a tree
- Start a fire at one site or on one edge
- How long does it take for a fire to burn out?
- How many trees are burned?

$p \approx p_c$

The burn-out time diverges at p_c!

- An example of singular behavior at the percolation transition
- Singularity is due to the connectivity of the infinite cluster at p_c



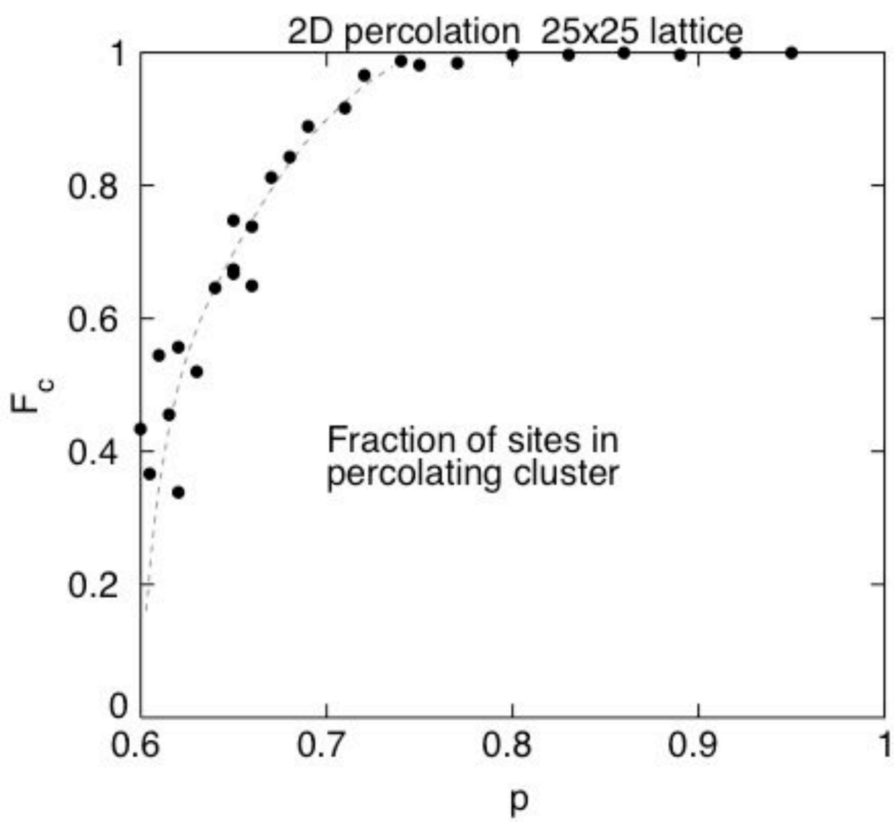
The spanning cluster is very tenuously connected

• The spanning cluster can be spoiled by removing only a few (1!) sites

$p \approx p_c$

Strange properties at p_c

- The spanning cluster is infinite (since it spans the system) but contains a vanishing fraction of the occupied sites!
- Forms a fractal



Focus on just the spanning (critical) cluster at p_c

- Remove all sites that are not part of the infinite cluster
- The spanning cluster contains large holes
- Need a way to describe the geometry of this cluster

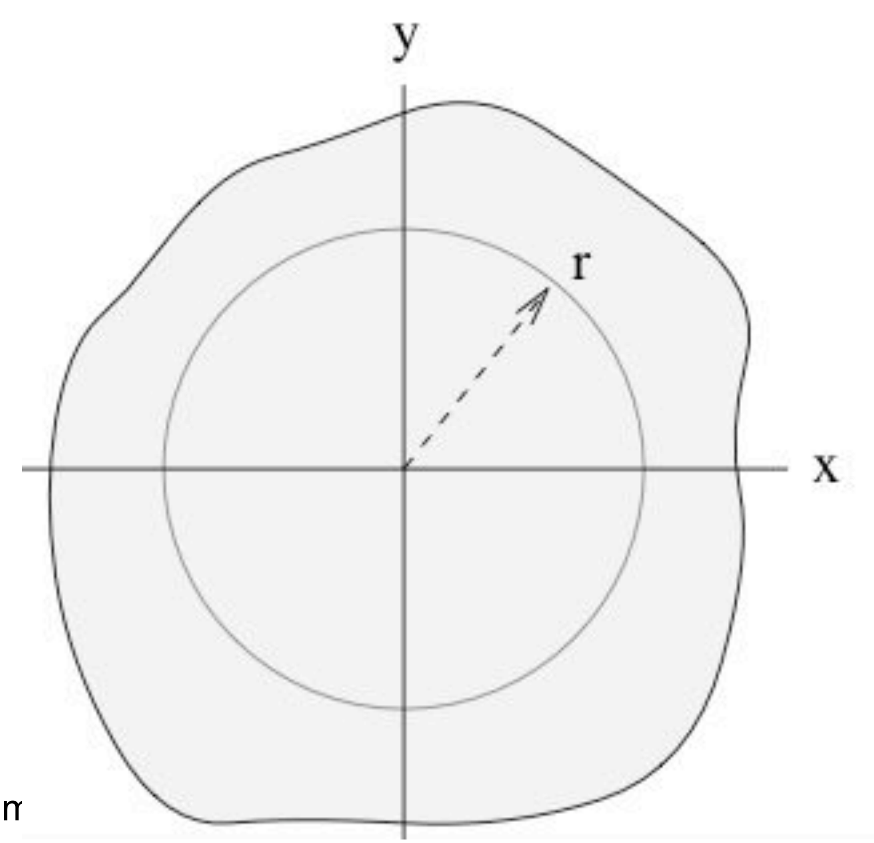
p = 0.60

Define the effective (fractal) dimensionality of a cluster

- Consider how the mass varies with *r*
- *m* varies as a power law

$$m(r) \sim r^{d_f}$$

- $d \sim r^2$ for a "regular 2-D cluster
- $d_f < 2$ for the spanning cluster at p_c
- => *fractal cluster*

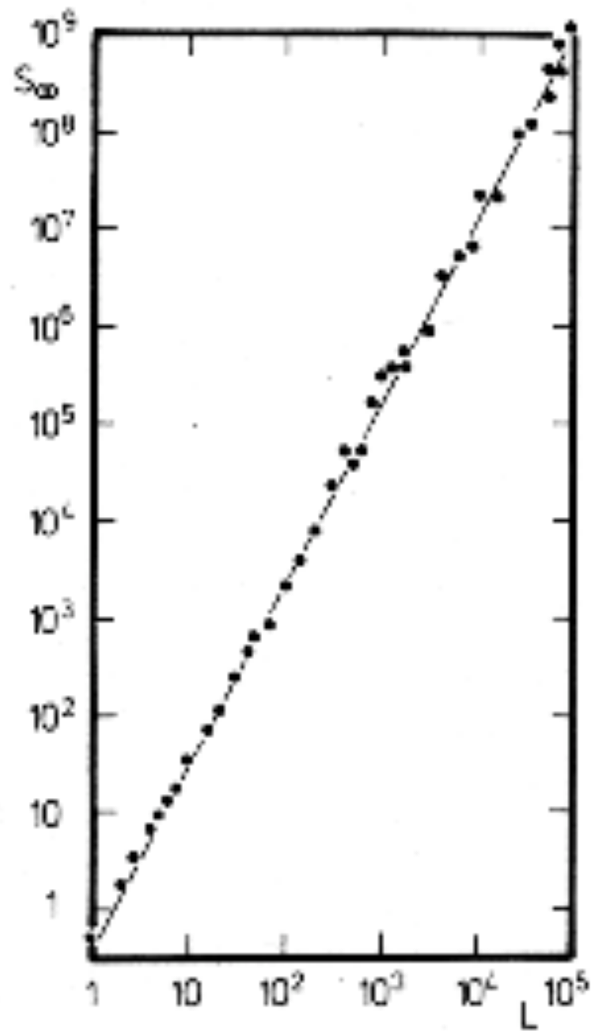


fractal scaling

- mass (m) of largest cluster as a function of lattice size (L)

$$m \sim r^{d_f}$$

- $d_f = 91/48 \approx 1.90$



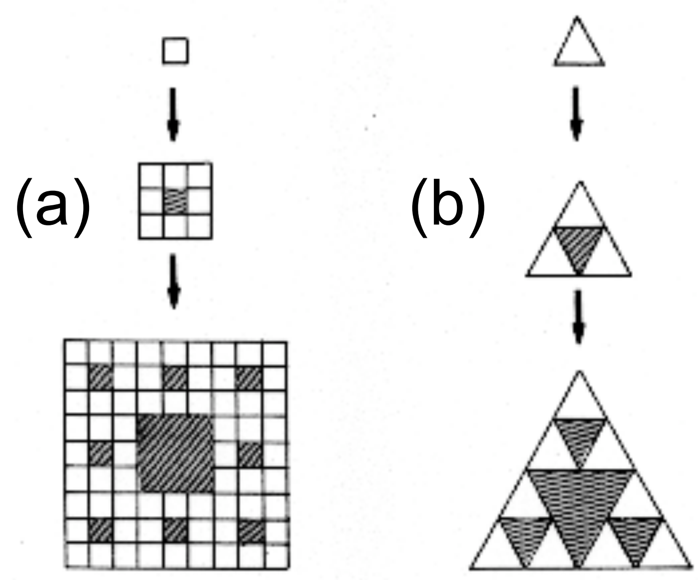
What makes a fractal cluster different?

- Just having holes and cracks is not enough
- Presence of “holes” and “cracks” on all length scales

p = 0.60

Can construct regular fractals using recursive algorithms

- Called Sierpinski "gaskets"
- Useful for analytic theory
- For cluster (a) exact d_f = log 8 / log 3 = 1.893



Consider properties

- Size of largest connected cluster
  - ➢ relevant to oil extraction
- Conductivity near p_c
  - ➢ most theory for electrical conductivity
  - ➢ can also consider fluid “conductivity”
- Mechanical properties
  - ➢ rigidity (Young’s modulus)
  - ➢ sound propagation

Properties of infinite cluster above p_c

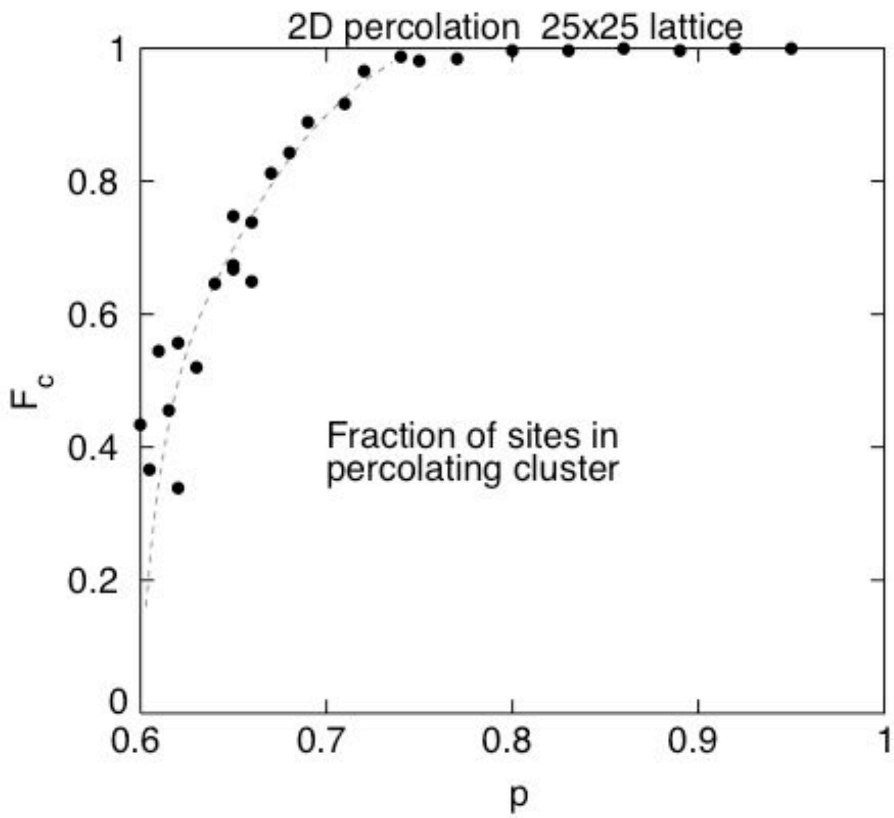
- fraction of sites in largest cluster

$$F \sim (p - p_c)^{\beta} \quad \beta \sim 5/36 \text{ (2D)}, \ 0.41 \text{ (3D)}$$

- size of largest cluster

$$s \sim (p - p_c)^{\xi}$$

$$\xi \sim 4/3 \text{ (2D)}, \ 0.88 \text{ (3D)}$$

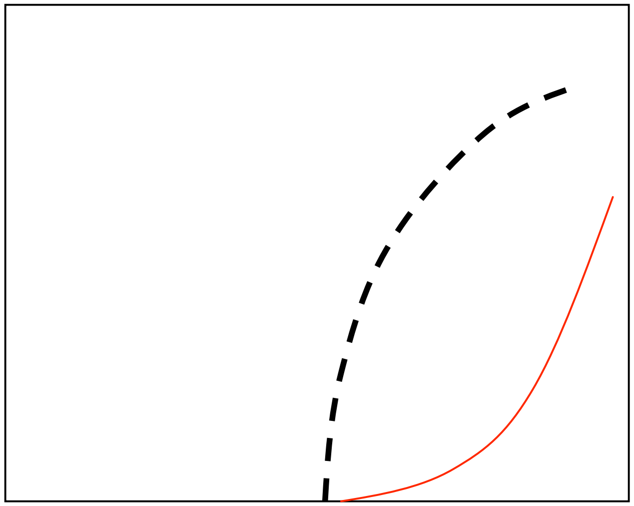


Conductivity vanishes at p_c

- Near p_c the conductivity vanishes as a power law

$$\sigma \sim (p - p_c)^{\mu} \to 0 \ \text{ at } p_c$$

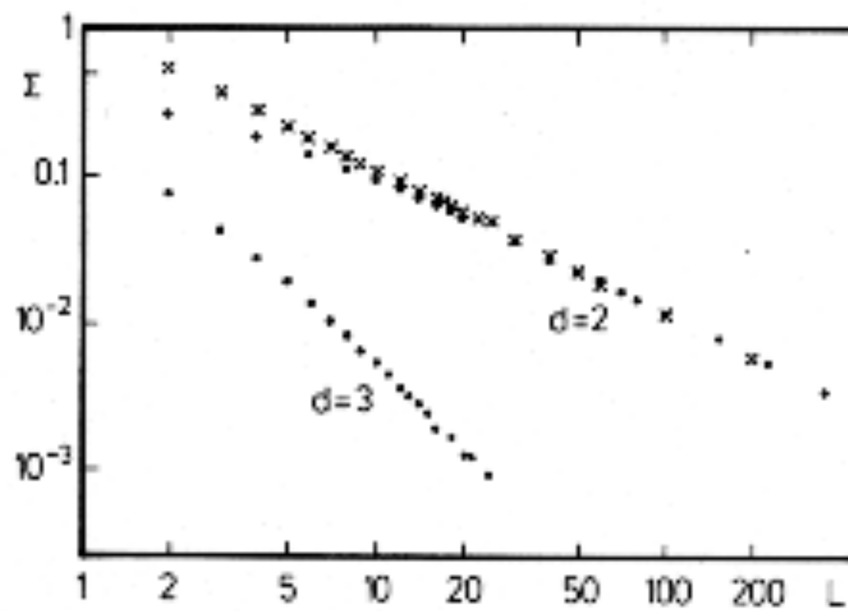
- μ = 1.30 (2D) 2.0 (3D)
- different behavior than cluster properties



Scaling of the electrical conductivity with system size at p_c

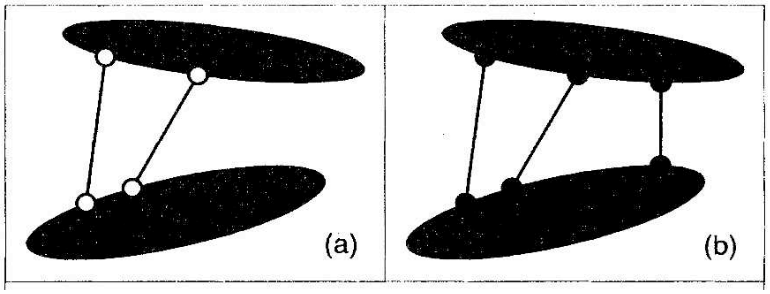
$$\sigma \sim (L - L_c)^{\mu/\nu} \rightarrow 0 \ \text{ at } p_c$$

- Exponents are not independent



Elastic properties

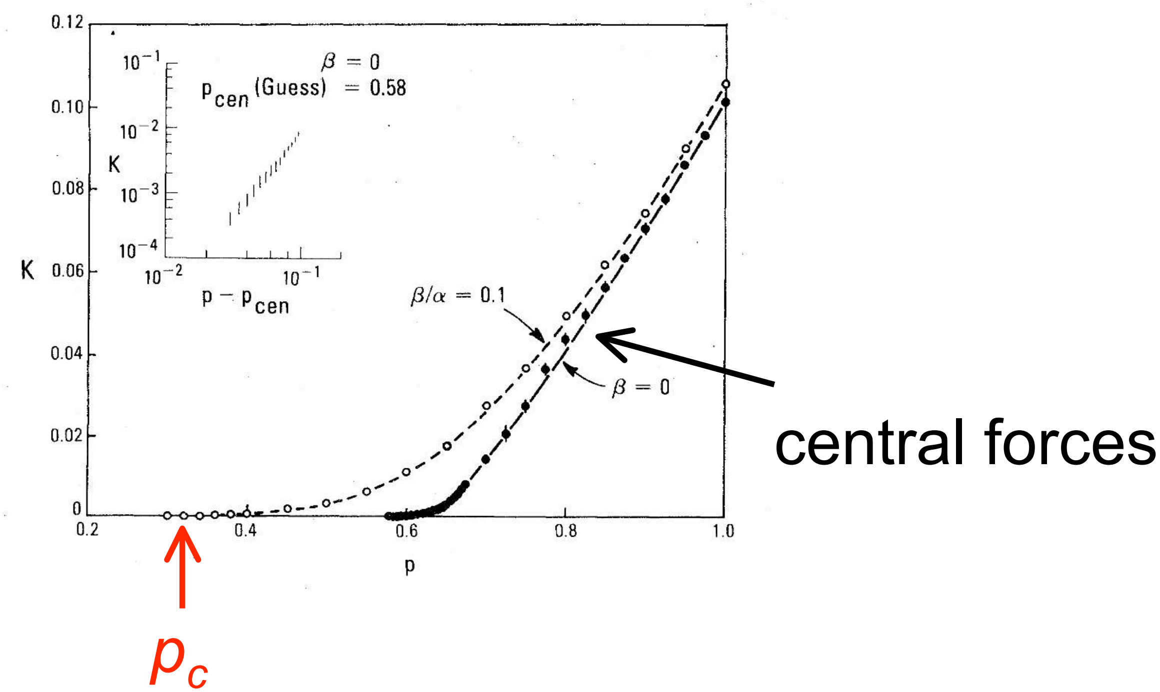
- System can be "floppy" (shear modulus = 0) even above p_c



- "Rigidity" threshold can be above p_c!
- Bonding bending forces move transition back to p_c but behavior is still complicated

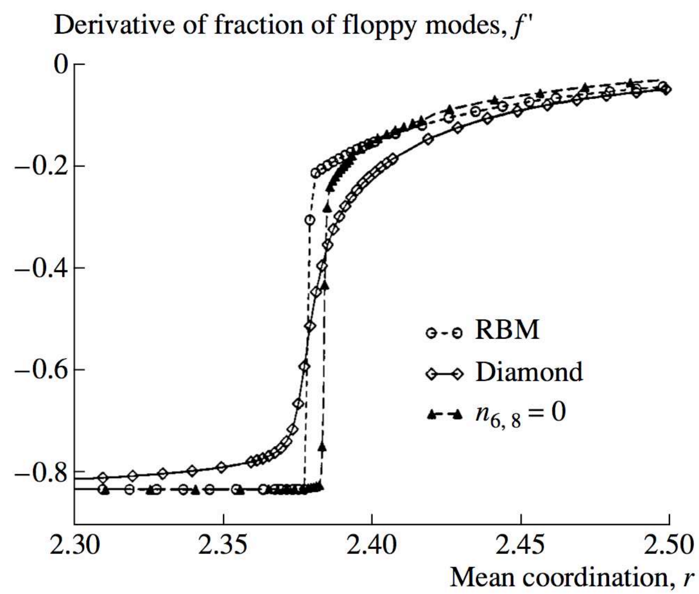
Behavior of elastic moduli above p_c

- with purely central forces (no bond bending) elastic constants go to zero above p_c
- with bond bending get crossover behavior



"First order"-like behavior

- f = fraction of floppy modes
- in some cases f' is discontinuous -- a first order transition



Open issues

- Properties away from p_c may be of greatest interest
  - ➢ we shouldn't focus only on p_c
- Real systems may not be truly random
  - ➢ must consider how they are made
  - ➢ etching or erosion of a solid will have a different p_c than a randomly occupied system
  - ➢ cracks "propagate" and spread

Summary

- Percolation is a type of phase transition
- Singular behavior at p_c
  - ➢ characterized by critical exponents
  - ➢ exponents depend on property and dimensionality
- Elastic properties very interesting
  - ➢ can affect elastic moduli and sound propagation
- Real percolative media can be more complicated
  - ➢ how system is produced affects geometry

References

- General reference:
  - ➢ D. Stauffer and A. Aharony, Introduction to Percolation Theory, 2nd edition (Taylor and Francis, 1992)
- Rigidity percolation:
  - ➢ Feng and Sen, Phys Rev Lett 52, 216 (1984)
  - ➢ Jacobs and Thorpe, Phys Rev E53, 3682 (1996)
  - ➢ Thorpe, et al., J. Non-Crystalline Solids 266-269, 859 (2000)