Introduction to Percolation
N Giordano -- Purdue University

• What is percolation?
• The percolation threshold - connection with phase transitions and critical phenomena
• Fractals and fractal scaling
  ➤ upscaling from small to large scales
• Properties
  ➤ conductivity
  ➤ fluid flow
  ➤ strength
• Open issues

[Recommended reference: Introduction to Percolation Theory, by Stauffer and Aharoni]
What is Percolation?

- Consider percolation on a lattice

  - Behavior depends on dimensionality (a lot) and lattice type (a little)
  - Can also consider continuum percolation (more realistic for us, but not covered in these lectures)

<p>| | |</p>
<table>
<thead>
<tr>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>square (2D)</td>
<td>honeycomb (2D)</td>
</tr>
</tbody>
</table>

July, 2006

Random media summer school
What is Percolation?

• Start with an empty lattice - then occupy sites at random

• Connected occupied sites form clusters

• Percolation is about the properties of these clusters -- size, connectivity, etc.
Consider connectivity across the lattice

- Connectivity depends on concentration of occupied sites $= \rho$
- Connectivity changes at $\rho_c \approx 0.59$ for site percolation on a square lattice

\[ \rho = 0.40 \quad \rho = 0.60 \quad \rho = 0.80 \]
$p_c$ is the “critical” concentration for percolation

- A “connectivity” phase transition occurs at $p_c \sim 0.59$
- A spanning cluster first appears at $p_c$
- Many properties are singular at $p_c$

$p = 0.40$ $p = 0.60$ $p = 0.80$
$p_c$ depends on lattice type

- $p_c$ is also different for site versus bond percolation

<table>
<thead>
<tr>
<th>Lattice</th>
<th>Site</th>
<th>Bond</th>
</tr>
</thead>
<tbody>
<tr>
<td>Honeycomb</td>
<td>0.6962</td>
<td>0.65271</td>
</tr>
<tr>
<td>Square</td>
<td>0.592746</td>
<td>0.50000</td>
</tr>
<tr>
<td>Triangular</td>
<td>0.50000</td>
<td>0.34729</td>
</tr>
<tr>
<td>Diamond</td>
<td>0.43</td>
<td>0.388</td>
</tr>
<tr>
<td>Simple cubic</td>
<td>0.3116</td>
<td>0.2488</td>
</tr>
<tr>
<td>BCC</td>
<td>0.246</td>
<td>0.1803</td>
</tr>
<tr>
<td>FCC</td>
<td>0.198</td>
<td>0.119</td>
</tr>
<tr>
<td>$d=4$ hypercubic</td>
<td>0.197</td>
<td>0.1601</td>
</tr>
<tr>
<td>$d=5$ hypercubic</td>
<td>0.141</td>
<td>0.1182</td>
</tr>
<tr>
<td>$d=6$ hypercubic</td>
<td>0.107</td>
<td>0.0942</td>
</tr>
<tr>
<td>$d=7$ hypercubic</td>
<td>0.089</td>
<td>0.0787</td>
</tr>
</tbody>
</table>
Why is $p_c$ special?

• Consider the forest fire problem
• Each occupied site is a tree
• Start a fire at one site or on one edge
• How long does it take for a fire to burn out?
• How many trees are burned?

$p \approx p_c$
The burn-out time diverges at $\rho_c$!

- An example of singular behavior at the percolation transition
- Singularity is due to the connectivity of the infinite cluster at $\rho_c$
The spanning cluster is very tenuously connected

- The spanning cluster can be spoiled by removing only a few (1!) sites

\[ p \approx p_c \]
Strange properties at $\rho_c$

- The spanning cluster is infinite (since it spans the system) but contains a vanishing fraction of the occupied sites!
- Forms a fractal
Focus on just the spanning (critical) cluster at $\rho_c$

- Remove all sites that are not part of the infinite cluster
- The spanning cluster contains large holes
- Need a way to describe the geometry of this cluster
Define the effective (fractal) dimensionality of a cluster

• Consider how the mass varies with $r$
• $m$ varies as a power law:

$$m(r) \sim r^{d_f}$$

• $d \sim r^2$ for a "regular 2-D cluster"
• $d_f < 2$ for the spanning cluster at $p_c$
• $\Rightarrow$ fractal cluster
fractal scaling

• mass ($m$) of largest cluster as a function of lattice size ($L$)

\[ m \sim r^{d_f} \]

• \[ d_f = \frac{91}{48} \approx 1.90 \]
What makes a fractal cluster different?

• Just having holes and cracks is not enough
• Presence of “holes” and “cracks” on all length scales

$p = 0.60$
Can construct regular fractals using recursive algorithms

- Called Sierpinski “gaskets”
- Useful for analytic theory
- For cluster (a) exact $d_f = \log 8 / \log 3 = 1.893$
Consider properties

- Size of largest connected cluster
  - relevant to oil extraction
- Conductivity near $p_c$
  - most theory for electrical conductivity
  - can also consider fluid “conductivity”
- Mechanical properties
  - rigidity (Young’s modulus)
  - sound propagation
Properties of infinite cluster above $p_c$

- fraction of sites in largest cluster
  \[ F \sim (p - p_c)^\beta \quad \beta \sim 5/36 \text{ (2D)}, \, 0.41 \text{ (3D)} \]

- size of largest cluster
  \[ s \sim (p - p_c)^\xi \]
  \[ \xi \sim 4/3 \text{ (2D)}, \, 0.88 \text{ (3D)} \]
Conductivity vanishes at $p_c$

- Near $p_c$ the conductivity vanishes as a power law
  
  \[ \sigma \sim (p - p_c)^\mu \rightarrow 0 \text{ at } p_c \]

- $\mu = 1.30 \ (2D) \ 2.0 \ (3D)$
- different behavior than cluster properties

\[ p_c \]
Scaling of the electrical conductivity with system size at $p_c$

$$\sigma \sim (L - L_c)^{\mu/\nu} \rightarrow 0 \text{ at } p_c$$

- Exponents are not independent
Elastic properties

- System can be “floppy” (shear modulus = 0) even above $p_c$

- “Rigidity” threshold can be above $p_c$!
- Bonding bending forces move transition back to $p_c$ but behavior is still complicated
Behavior of elastic moduli above $p_c$

- with purely central forces (no bond bending) elastic constants go to zero above $p_c$
- with bond bending get crossover behavior
“First order”-like behavior

- $f =$ fraction of floppy modes
- in some cases $f'$ is discontinuous -- a first order transition
Open issues

• Properties away from $p_c$ may be of greatest interest
  ➤ we shouldn’t focus only on $p_c$

• Real systems may not be truly random
  ➤ must consider how they are made
  ➤ etching or erosion of a solid will have a different $p_c$ than a randomly occupied system
  ➤ cracks “propagate” and spread
Summary

• Percolation is a type of phase transition
• Singular behavior at $p_c$
  ➢ characterized by critical exponents
  ➢ exponents depend on property and dimensionality
• Elastic properties very interesting
  ➢ can affect elastic moduli and sound propagation
• Real percolative media can be more complicated
  ➢ how system is produced affects geometry
References

• General reference:

• Rigidity percolation: