SOME THEOREMS IN POTENTIAL THEORY

We have already established some elementary results, such as

$$\vec{F} = -\vec{\nabla} V \Leftrightarrow \vec{\nabla} \times \vec{F} = 0 \qquad (1)$$

$$\vec{B} = \vec{\nabla} \times \vec{A} \Leftrightarrow \vec{\nabla} \cdot \vec{B} = 0 \qquad (2)$$

To establish some other results we require a discussion of the Dirac δ-function.

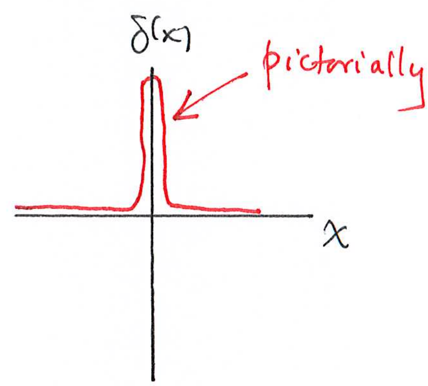
REVIEW OF THE DIRAC δ-FUNCTION

In 1-dimension $\delta(x)$ is defined by

$$\boxed{\int_{-\infty}^{\infty} dx\, \delta(x) f(x) = f(0)} \qquad (3)$$

Equivalently, define $\delta(x)$ by

$$\left.\begin{array}{l} \delta(x) = 0 \quad x \neq 0 \\ \int_{-\infty}^{\infty} dx\, \delta(x) = 1 \end{array}\right\} \quad (4)$$



IMPORTANT!! $\delta(x)$ is not an ordinary mathematical function such as e^{-x}. A relation such as (3) must be understood as holding when $\delta(x)$ is integrated along with a smooth convergent test function such as e^{-x^2}, which vanishes as $x \to \pm\infty$. With this understanding the following relations hold when appearing under an integral with $f(x)$:

Useful Relations Involving $\delta(x)$:

$$
\begin{aligned}
&(a)\quad \delta(x) = \delta(-x) \\
&(b)\quad \delta'(x) = -\delta'(-x) \\
&(c)\quad x\delta(x) = 0 \\
&(d)\quad x\,\delta'(x) = -\delta(x) \\
&(e)\quad \delta(ax) = \frac{1}{|a|}\delta(x) \quad ; \; a = \text{constant} \\
&(f)\quad \delta(x^2-a^2) = \frac{1}{|2a|}\left[\delta(x-a)+\delta(x+a)\right] \\
&(g)\quad \int dx\ \delta(x-a)\delta(x-b) = \delta(a-b) \\
&(h)\quad f(x)\,\delta(x-a) = f(a)\,\delta(x-a) \\
&(i)\quad \delta[g(x)] = \sum_i \frac{1}{|\partial g/\partial x|_{x_i}}\,\delta(x-x_i) \; ;
\end{aligned}
\tag{5}
$$

$\boxed{g(x_i)=0\text{, so } x_i \text{ are the roots of } g(x)}$

You will be asked to establish these for homework; here we illustrate with 2 examples:

(5d): $$\int_{-\infty}^{\infty} dx\,[x\delta'(x)]f(x) \equiv \int_{-\infty}^{\infty} dx\,x\left[\frac{d}{dx}\delta(x)\right]f(x) = \int_{-\infty}^{\infty} dx\ \underbrace{xf(x)}_{u}\cdot\underbrace{\frac{d}{dx}\delta(x)}_{dV} \tag{6}$$

$$= \cancel{xf(x)\delta(x)}\Big|_{-\infty}^{\infty} - \int_{-\infty}^{\infty} dx\cdot\delta(x)\cdot\underbrace{\frac{d}{dx}[xf(x)]}_{f(x)+xf'(x)} = -\int_{-\infty}^{\infty} dx\ \delta(x)f(x) - \int_{-\infty}^{\infty} dx\cdot \underbrace{x\delta(x)}_{0} f'(x)$$

Hence: $$\int_{-\infty}^{\infty} dx\,[x\delta'(x)]f(x) = \int_{-\infty}^{\infty} dx\ [-\delta(x)]f(x) \Rightarrow \underbrace{x\delta'(x) \sim -\delta(x)}_{\text{under an integral!!}} \tag{7}$$

(5i) This can be proved by noting that $\delta[g(x)]$ will differ from zero only when $g(x)=0$ which means that this holds for values $x=x_i$ which are the roots of $g(x)$: $g(x_i)=0$.

Hence in the vicinity of each root we can expand $g(x)$ as

$$g(x) = \underbrace{g(x_i)}_{0} + (x-x_i)\left.\frac{dg}{dx}\right|_{x_i} + \frac{1}{2}(x-x_i)^2 \left.\frac{d^2g}{dx^2}\right|_{x_i} + \cdots \qquad (8)$$

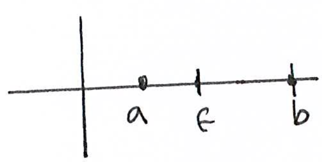
$$= (x-x_i)\left\{\left.\frac{dg}{dx}\right|_{x_i} + \underbrace{\frac{1}{2}(x-x_i)\left.\frac{d^2g}{dx^2}\right|_{x_i} + \cdots}_{0}\right\} \cong (x-x_i)\underbrace{\left.\frac{dg}{dx}\right|_{x_i}}_{\text{constant} \equiv a} \qquad (9)$$

Hence near a root x_i:

$$\delta[g(x)] \cong \delta\left[\left.\frac{dg}{dx}\right|_{x_i}(x-x_i)\right]$$

$$= \frac{1}{\left|\partial g/\partial x\right|_{x_i}}\,\delta(x-x_i) \leftarrow \text{using 5(e)} \qquad (10)$$

Since we can repeat this process for each root, we sum over all the roots. This can be seen from the following example: Consider the function $g(x) = (x-a)(x-b)$ with roots at $x=a$ and $x=b$. Given $g(x)$ we have



$$\int_{-\infty}^{\infty} dx\,\delta[g(x)]f(x) = \underbrace{\int_{-\infty}^{\epsilon} dx\,\delta[(x-a)(x-b)]f(x)}_{\text{I}} + \underbrace{\int_{\epsilon}^{+\infty} dx\,\delta[(x-a)(x-b)]f(x)}_{\text{II}} \qquad (11)$$

Near $x=a$ (I) gives: (I) $\cong f(a)\int_{-\infty}^{\epsilon} dx\,\delta[(x-a)(a-b)] = \frac{f(a)}{|a-b|}\underbrace{\int_{-\infty}^{\epsilon} dx\,\delta(x-a)}_{1}$ (12)

$$= \frac{1}{|a-b|}f(a)$$

Near $x=b$ (II) gives: (II) $\cong f(b)\int_{\epsilon}^{\infty} dx\,\delta[(b-a)(x-b)] = \frac{f(b)}{|b-a|}$ (13)

Combining the results in (11)–(13) we have

$$\int_{-\infty}^{\infty} dx\, \delta[g(x)]\, f(x) = \frac{1}{|a-b|} f(a) + \frac{1}{|b-a|} f(b) = \frac{1}{|a-b|}\left[f(a) + f(b)\right] \qquad (14)$$

Compare this to the result using the formula in (5i):

$$dg/dx = d/dx\left(x^2 - (a+b)x + ab\right) = 2x - (a+b) \qquad (15)$$

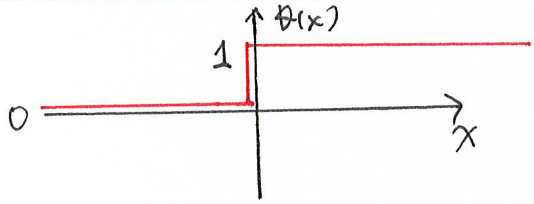
$$\partial g/\partial x\big|_{x=a} = 2a - (a+b) = a - b \qquad (16)$$

$$\partial g/\partial x\big|_{x=b} = 2b - (a-b) = b - a$$

Hence
$$\delta[g(x)] = \sum_i \frac{1}{|\partial g/\partial x|_{x_i}}\, \delta(x - x_i) = \frac{1}{|a-b|}\delta(x-a) + \frac{1}{|b-a|}\delta(x-b) \qquad (17)$$

and this clearly reproduces (14) above. ✓

The Step Function $\theta(x)$:



$$\theta(x) = 1 \,;\; x > 0$$
$$\theta(x) = 0 \,;\; x < 0$$
$$\theta(0) \equiv 1/2$$

Claim: $$\frac{d}{dx}\theta(x) = \delta(x) \qquad (1)$$

Proof: Consider $I = \int_{-\infty}^{\infty} dx \left[\frac{d}{dx}\theta(x)\right] f(x)$ where $f(\pm\infty) = 0$

Then $$\int_{-\infty}^{\infty} dx \underbrace{\left[\frac{d}{dx}\theta(x)\right]}_{dv} \underbrace{f(x)}_{u} = \theta(x) f(x)\Big|_{-\infty}^{\infty} - \int_{-\infty}^{\infty} dx\, \theta(x) \frac{d}{dx} f(x) \qquad (2)$$

$$= -\int_{0}^{\infty} \cancel{dx} \frac{d}{\cancel{dx}} f(x) = -\int_{0}^{\infty} d f(x) = -[\underbrace{f(\infty)}_{=0} - f(0)] = +f(0) \qquad (3)$$

Comparing in (2) & (3) we see that $\left[\frac{d}{dx}\theta(x)\right]$ has the same effect as $\delta(x)$. ✓

Specific Representations of $\delta(x)$:

As noted previously, $\delta(x)$ is not a conventional mathematical function. Rather it can be viewed as the limiting case of a function whose width decreases as its height increases (when some parameter is varied) in such a way that its area remains $= 1$.

We present several examples:

(A) $$\boxed{f_a(x) \equiv \frac{1}{a\sqrt{\pi}} e^{-x^2/a^2}} \; ; \; \int_{-\infty}^{\infty} dx\, f_a(x) = 1 \quad \underline{\text{independent of } a} \qquad (1)$$

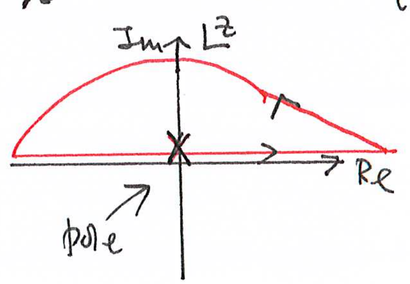
Then $$\boxed{\delta(x) = \lim_{a\to 0} f_a(x) = \lim_{a\to 0} \frac{1}{a\sqrt{\pi}} e^{-x^2/a^2}} \qquad (2)$$

Here we note that as $a \to 0$ $\quad e^{-x^2/a^2} \to 0$ for $x \neq 0$; moreover $e^{-x^2/a^2} \to 0$ faster than $1/a \to \infty$. Hence $f_a(x \neq 0) = 0$ as $a \to 0$. However, as $a \to 0$ $f_a(0) \sim \infty$ to keep the area constant.

(B) $$\boxed{h_g(x) = \frac{\sin gx}{\pi x}} \qquad (3)$$

This can be integrated using contour integration (see end of semester!)

$$\int_{-\infty}^{\infty} dx\, h_g(x) = \operatorname{Im} \frac{1}{\pi} \int_{-\infty}^{\infty} dx\, \frac{1}{x} e^{igx} = \frac{1}{\pi} \operatorname{Im}\left\{ \pi i \left[e^{igx} \right]_{x=0} \right\} = 1 \qquad (4)$$



We note that for $x \cong 0$ $h_g(x) \cong \frac{g}{\pi}$; Hence as $g \to \infty$ $h_g(x \cong 0) \to \infty$

Since $\int_{-\infty}^{\infty} dx\, h_g(x) = 1$ (for all values of g) it follows that | 28, 29

the remaining contributions for $x \neq 0$ are becoming vanishingly small.
This happens because $\sin(gx)$ oscillates very rapidly as $g \to \infty$
(this is the Riemann-Lebesgue Theorem). We can thus finally
write:

$$\boxed{\delta(x) = \lim_{g \to \infty} \frac{\sin gx}{\pi x}} \quad (5)$$

(c) The 3RD representation that we consider is

$$\boxed{\delta(x) = \lim_{\epsilon \to 0} \frac{1}{\pi} \frac{\epsilon}{x^2+\epsilon^2}} \quad (6)$$

Clearly the r.h.s. of (6) vanishes as $\epsilon \to 0$ for all $x \neq 0$. For
$x = 0$ the r.h.s. $\to 1/\epsilon$ as $\epsilon \to 0$, so (6) has the correct behavior.

Note that
$$\int_{-\infty}^{\infty} dx\, \delta(x) = \lim_{\epsilon \to 0} \frac{\epsilon}{\pi} \int_{-\infty}^{+\infty} dx \frac{1}{x^2+\epsilon^2} = \lim_{\epsilon \to 0} \frac{\epsilon}{\pi} \cdot \frac{1}{\epsilon} \tan^{-1} \frac{x}{\epsilon} \Big|_{-\infty}^{\infty}$$
$$= \frac{1}{\pi}\left(\frac{\pi}{2} - \left(-\frac{\pi}{2}\right)\right) = 1 \; ; \; \underline{\underline{\text{independent of } \epsilon}} \quad (7)$$

Hence the function in (6) also has unit area (independent of ϵ),
and as $\epsilon \to 0$ this function vanishes everywhere except at $x = 0$.
From the previous discussion this establishes that (6) is a valid
representation of $\delta(x)$.

Comments on Representations of $\delta(x)$:

Here we evaluate some of the integrals we discussed previously.

Consider
$$I = \int_{-\infty}^{\infty} dx\, e^{-x^2} \Rightarrow I^2 = \int_{-\infty}^{\infty} dx\, e^{-x^2} \int_{-\infty}^{\infty} dy\, e^{-y^2} \qquad (1)$$

Hence
$$I^2 = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} dx\,dy\; e^{-(x^2+y^2)} \qquad (2)$$

Transforming to polar coordinates: $dx\,dy \to 2\pi r\,dr \qquad x^2+y^2=r^2$

$$\therefore\ I^2 = 2\pi \int_0^{\infty} dr \cdot r e^{-r^2} \xrightarrow[\rho=r^2]{} 2\pi\cdot\frac{1}{2}\int_0^{\infty} d\rho\, e^{-\rho} = -\pi e^{-\rho}\Big|_0^{\infty} = \pi \qquad (3)$$

$\Rightarrow d\rho = 2r\,dr$

Hence $I^2 = \pi \Rightarrow$
$$\boxed{I = \int_{-\infty}^{\infty} dx\, e^{-x^2} = \sqrt{\pi}} \qquad (4)$$

It follows from (4) that
$$\int_{-\infty}^{\infty} dy\, e^{-y^2/a^2} = a\int_{-\infty}^{\infty} dx\, e^{-x^2} = a\sqrt{\pi} \qquad (5)$$

Hence
$$\boxed{\frac{1}{a\sqrt{\pi}}\int_{-\infty}^{\infty} dy\, e^{-y^2/a^2} = 1\ ;\ \underline{\text{independent of } a}} \qquad (6)$$

Other related integrals can be evaluated in a similar way: Consider

$$I^3 = (\sqrt{\pi})^3 = \iiint_{-\infty}^{\infty} dx\,dy\,dz\; e^{-(x^2+y^2+z^2)} = 4\pi\int_0^{\infty} dr \cdot r^2 e^{-r^2} \qquad (7)$$

Hence
$$\boxed{\int_0^{\infty} dr \cdot r^2 e^{-r^2} = \frac{\sqrt{\pi}}{4}} \qquad (8)$$

Another way to derive Eq. (8) is to start with Eq. (6) and let $b = 1/a^2$. Then

$$f(b) \equiv \int_{-\infty}^{\infty} dy \; e^{-by^2} = \sqrt{\frac{\pi}{b}} \qquad (9)$$

$$\frac{d f(b)}{db} = -\int_{-\infty}^{\infty} dy \cdot y^2 e^{-by^2} = \frac{d}{db}\left(\sqrt{\frac{\pi}{b}}\right) = -\frac{1}{2}\sqrt{\frac{\pi}{b^3}} \qquad (10)$$

Combining (9) & (10) we find: $\boxed{\int_{0}^{\infty} dy \; y^2 e^{-by^2} = \frac{1}{2}\int_{-\infty}^{\infty} dy \ldots = \frac{1}{4}\sqrt{\frac{\pi}{b^3}}}$ (11)

Setting $b=1$ in (11) then leads immediately to (8). ✓

KEY THEOREM IN POTENTIAL THEORY:

If the divergence and curl of a vector field $\vec{V}(r)$ are known,

$$\vec{\nabla}\cdot\vec{V}(\vec{r}) = s(\vec{r}) \quad \sim \text{charges} \qquad (1a)$$

$$\vec{\nabla}\times\vec{V}(\vec{r}) = \vec{c}(\vec{r}) \quad \sim \text{currents} \qquad (1b)$$

throughout space, and if there are no sources or currents at ∞ $[\,s(\infty)=0 \quad \vec{c}(\infty)=0\,]$ then $\vec{V}(\vec{r})$ is uniquely given by

$$\boxed{\vec{V}(\vec{r}) = -\vec{\nabla}\phi(\vec{r}) + \vec{\nabla}\times\vec{A}(\vec{r})} \qquad (2)$$

where $(\vec{x}\equiv\vec{r})$

$$\phi(\vec{x}) = \frac{1}{4\pi}\int d^3x' \, \frac{s(\vec{x}')}{r(\vec{x},\vec{x}')} \qquad (3)$$

$$\vec{A}(\vec{x}) = \frac{1}{4\pi}\int d^3x' \, \frac{\vec{c}(\vec{x}')}{r(\vec{x},\vec{x}')} \quad ; r(\vec{x},\vec{x}') = \sqrt{(x-x')^2+(y-y')^2+(z-z')^2} \qquad (4)$$

Proof: We first show that (2) is a solution, and then show that it is the unique solution.

Consider first $\vec{\nabla}\cdot\vec{V}(\vec{x}) = -\nabla^2\phi(\vec{x}) + \underbrace{\vec{\nabla}\cdot[\vec{\nabla}\times\vec{A}(\vec{x})]}_{=0 \text{ using } 17(8)}$ (5)

Hence $\vec{\nabla}\cdot\vec{V}(\vec{x}) = -\frac{1}{4\pi}\int d^3x' \, \nabla^2_{(x)}\left\{\frac{s(\vec{x}')}{r(\vec{x},\vec{x}')}\right\}$ (6)

this means that ∇^2 acts only on $\vec{x}$ and not on $\vec{x}'$ in $\{\cdots\}$

Then

$$\nabla^2\left(\frac{1}{r}\right) = \nabla^2\left(\frac{1}{\sqrt{(x-x')^2+(y-y')^2+(z-z')^2}}\right) = \underbrace{-4\pi\delta^3(\vec{r})}_{20.1(1)} = -4\pi\delta^3(\vec{x}-\vec{x}') \qquad (7)$$

Hence $\vec{\nabla}\cdot\vec{V}(\vec{x}) = -\nabla^2\phi(\vec{x}) = -\frac{1}{4\pi}\int d^3x' \left\{-4\pi\delta^3(\vec{x}-\vec{x}')\,s(\vec{x}')\right\}$ (8)

$$\boxed{\therefore\ \vec{\nabla}\cdot\vec{V}(\vec{x}) = s(\vec{x})\ \checkmark} \qquad (9)$$

This establishes that Eq. (2) is indeed the solution to Eq. (1a).

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We next show that Eq. (2) is also the solution to Eq. (1b). This requires some more effort, but allows us to gain some practice manipulating ∇^2, $\vec{\nabla}\times$, ...

From (2) we have: $\vec{\nabla}\times\vec{V} = \vec{\nabla}\times\left\{-\vec{\nabla}\phi + \vec{\nabla}\times\vec{A}\right\}$ (10)

$$= \underbrace{-\vec{\nabla}\times(\vec{\nabla}\phi)}_{0} + \vec{\nabla}\times(\vec{\nabla}\times\vec{A}) \underset{23(17)}{=} -\nabla^2\vec{A} + \vec{\nabla}(\vec{\nabla}\cdot\vec{A})$$

Hence $\vec{\nabla}\times\vec{V}(\vec{x}) = -\frac{1}{4\pi}\int d^3x'\, \nabla^2_{(x)}\left\{\frac{\vec{c}(\vec{x}')}{r(\vec{x},\vec{x}')}\right\}$ (11) $\to$ I

$$+\frac{1}{4\pi}\int d^3x'\, \vec{\nabla}_{(x)}\left\{\vec{\nabla}_{(x)}\cdot\left(\frac{\vec{c}(\vec{x}')}{r(\vec{x},\vec{x}')}\right)\right\} \to \text{II}$$

We will later show that II $= 0$. Assuming this for now we can directly repeat the steps leading to (9) which then give from I

$$\vec{\nabla}\times\vec{V}(\vec{x}) = -\frac{1}{4\pi}\int d^3x'\left[\nabla^2_{(x)}\left(\frac{1}{r}\right)\right]\vec{c}(\vec{x}') = -\frac{1}{4\pi}\int d^3x'\left(-4\pi\delta^3(\vec{x}-\vec{x}')\right)\vec{c}(\vec{x}') \qquad (12)$$

$$\therefore\ \boxed{\vec{\nabla}\times\vec{V}(x) = c(\vec{x})\ \checkmark} \qquad (13)$$

This establishes that (2) is also a solution of Eq. (1b), provided that we can now show that II $= 0$. 36,37

Define II $\equiv \vec{D} = \dfrac{1}{4\pi}\int d^3x'\, \vec{\nabla}_{(x)} \left[\vec{\nabla}_{(x)}\cdot \dfrac{\vec{c}(\vec{x}')}{r(\vec{x},\vec{x}')}\right]$ (14)

To clarify the following steps we insert subscripts on $\vec{\nabla}$ so that we can keep track of them. Both $\vec{\nabla}_{(x)}$ operators only operate on $\vec{x}$:

$$\vec{D} = \frac{1}{4\pi}\int d^3x'\, \vec{\nabla}_1\left[\vec{\nabla}_2\cdot\left(\frac{\vec{c}}{r}\right)\right] = \frac{1}{4\pi}\int d^3x'\, \vec{\nabla}_1\left[\vec{c}(\vec{x}')\cdot\vec{\nabla}_2\left(\frac{1}{r}\right)\right] \quad (15)$$

$$= \frac{1}{4\pi}\int d^3x'\,(\vec{c}\cdot\vec{\nabla}_2)(\vec{\nabla}_1(1/r)) \equiv \frac{1}{4\pi}\int d^3x'\,(\vec{c}\cdot\vec{\nabla})(\vec{\nabla}(1/r)) \quad (16)$$

Note here that both $\vec{\nabla}$ operators only act on $1/r = 1/r(\vec{x},\vec{x}')$, since $1/r$ contains the only dependence on $\vec{x}$. This can be made clearer if we write $\vec{D}$ in the form

$$\vec{D} = \frac{1}{4\pi}\int d^3x'\left[\vec{c}(\vec{x}')\cdot\vec{\nabla}_{(x)}\right]\left[\vec{\nabla}_{(x)}\left(\frac{1}{r(\vec{x},\vec{x}')}\right)\right] \quad (17)$$

We next introduce the following <u>trick</u>: First we now will denote

$$\vec{\nabla}_{(x)} = \hat{\imath}\frac{\partial}{\partial x} + \cdots + \hat{k}\frac{\partial}{\partial z} \equiv \vec{\nabla} \quad (18)$$

$$\vec{\nabla}' = \hat{\imath}\frac{\partial}{\partial x'} + \hat{\jmath}\frac{\partial}{\partial y'} + \hat{k}\frac{\partial}{\partial z'} \quad (19)$$

<u>Trick:</u> $\boxed{\vec{\nabla}' g[r(\vec{x},\vec{x}')] = -\vec{\nabla} g[r(\vec{x},\vec{x}')]}$ (20)

<u>Example:</u> Let $g(r) = \frac{1}{2}c\,r^2 = \frac{1}{2}c\left[(x-x')^2+(y-y')^2+(z-z')^2\right]$ (21)

Then $\vec{\nabla} g(r) = [\hat{\imath}\partial_x + \hat{\jmath}\partial_y + \hat{k}\partial_z]\, g(r) = \frac{c}{2}[\hat{\imath}(2)(x-x') + \dots] = c\vec{r}$

Compare this to $\vec{\nabla}' g(r) = [\hat{\imath}\partial_{x'} + \hat{\jmath}\partial_{y'} + \hat{k}\partial_{z'}]\, g(r) = \frac{c}{2}[\hat{\imath}(2)(x-x')(-1) + \dots] = -c\vec{r}$ (22)

here is where the sign is coming from

This establishes the validity of the trick in (20).

Using (20) we then return to the expression for $\vec{D}$ in (17) and replace $\vec{\nabla} \to -\vec{\nabla}'$. Since we do this <u>twice</u>, there is no sign change:

$$\vec{D} = \frac{1}{4\pi}\int d^3x' \left[\vec{C}(\vec{x}')\cdot\vec{\nabla}'\right]\left[\vec{\nabla}'\left(\frac{1}{r(\vec{x},\vec{x}')}\right)\right] \qquad (23)$$

Consider one of the components of $\vec{D}$, D_α ($\alpha = 1, 2,$ or 3). We can integrate by parts using the identity

this acts on <u>everything</u> to the right

$$\int d^3x'\, \vec{\nabla}'\cdot\left\{\vec{C}(\vec{x}')\frac{\partial}{\partial x'_\alpha}\left(\frac{1}{r}\right)\right\} = \int d^3x' \left(\vec{\nabla}'\cdot\vec{C}\right)\frac{\partial}{\partial x'_\alpha}\left(\frac{1}{r}\right) + \int d^3x' \left[\vec{C}\cdot\vec{\nabla}'\right]\frac{\partial}{\partial x'_\alpha}\left(\frac{1}{r}\right) \qquad (24)$$

Comparing (23) & (24) we see that

$$4\pi\vec{D} = \underbrace{\int d^3x'\, \vec{\nabla}'\cdot\left\{\vec{C}(\vec{x}')\,\vec{\nabla}'\left(\frac{1}{r}\right)\right\}}_{\text{A}\ \equiv \vec{F}(\vec{x}')} - \underbrace{\int d^3x' \left[\vec{\nabla}'\cdot\vec{C}(\vec{x}')\right]\vec{\nabla}'\left(\frac{1}{r}\right)}_{\text{B}} \qquad (25)$$

Keep in mind that we are trying to show that $\vec{D} = 0$, so we begin by showing that (A) $= 0$. This is a general and very widely used argument! Write

$$\text{(A)} \equiv \int_V d^3x'\, \vec{\nabla}'\cdot\vec{F}(\vec{x}') \overset{\text{Gauss}}{=} \int_S d\vec{S}'\cdot\vec{F}(\vec{x}') \qquad (26)$$

Here we make the standard argument that if $\vec{F}(\vec{x}\,')$ depends on a source function $\vec{c}(\vec{x}\,')$ which is localized in space Then a Gaussian surface S can be found (taking S large enough!) so that no flux from $\vec{c}(\vec{x}\,')$ crosses S, and hence Ⓐ $\equiv 0$.

[A similar argument is often used for the 4-dimensional version of Gauss' theorem, but care must be used there, since sources are not always localized in time!!]

Since Ⓐ $\equiv 0$, the combination of Eqs. (11), (13), (14), & (25) give

$$\vec{\nabla}\times\vec{V}(\vec{x}) = \vec{c}(\vec{x}) - \frac{1}{4\pi}\int d^3x' \left[\vec{\nabla}'\cdot\vec{c}(\vec{x}\,')\right]\vec{\nabla}'\left(\frac{1}{r}\right) \qquad (27)$$

We are trying to show that $\vec{\nabla}\times\vec{V}(\vec{x}) = \vec{c}(\vec{x})$; this follows by noting that Eq. (27) would hold if we replace $\vec{c}(\vec{x}\,')$ by $\vec{\nabla}'\times\vec{V}(\vec{x}\,')$. This is a self-consistency argument; We conclude from (27) that in fact $\vec{\nabla}\times\vec{V}(x) = \vec{c}(\vec{x})$, where $\vec{V}(\vec{x})$ is given by (2).

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Uniqueness of Solutions:

Question: Having shown that $\vec{\nabla}\cdot\vec{V} = S$ and $\vec{\nabla}\times\vec{V} = \vec{C}$, where $\vec{V}$ is given by (2), we now ask whether there can be 2 solutions $\vec{V}_1$ and $\vec{V}_2$ which work? Specifically can we find $\vec{V}_{1,2}$ such that

$$\boxed{\vec{\nabla}\cdot\vec{V}_{1,2}(\vec{x}) = S(\vec{x}) \quad \text{and} \quad \vec{\nabla}\times\vec{V}_{1,2}(\vec{x}) = \vec{c}(\vec{x})} \qquad (28)$$

Consider $\vec{W}(\vec{x}) = \vec{V}_1(\vec{x}) - \vec{V}_2(\vec{x})$. We want to show that $\vec{W}(\vec{x}) \equiv 0$.

From (28)

$$\vec{\nabla}\cdot\vec{W}(\vec{x}) = \vec{\nabla}\cdot\left[\vec{V}_1(\vec{x}) - \vec{V}_2(\vec{x})\right] = S(\vec{x}) - S(\vec{x}) = 0 \qquad (29)$$

$$\vec{\nabla}\times\vec{W}(\vec{x}) = \vec{\nabla}\times\left[\vec{V}_1(\vec{x}) - \vec{V}_2(\vec{x})\right] = \vec{c}(\vec{x}) - \vec{c}(\vec{x}) = 0$$

Since $\vec{\nabla}\times\vec{W}=0$ it follows from 18.3 (16) that $\vec{W}$ can be expressed as

$$\vec{W}=-\vec{\nabla}\psi \quad \leftarrow \text{Scalar field} \tag{30}$$

Then $\vec{\nabla}\cdot\vec{W}=0 \Rightarrow$ $\boxed{\nabla^2\psi(x)=0 \text{ everywhere}}$ (31)

To proceed using (31) we apply Gauss' theorem to the vector $\psi\vec{\nabla}\psi$:

$$\int \psi\vec{\nabla}\psi\cdot d\vec{S} = \int \vec{\nabla}\cdot(\psi\vec{\nabla}\psi)\,d\mathcal{V} \equiv \int \partial_i(\psi\partial_i\psi)\,d\mathcal{V} \tag{31}$$

← volume element

$$= \int[(\partial_i\psi)(\partial_i\psi)+\psi\partial_i\partial_i\psi]\,d\mathcal{V} = \int(\vec{\nabla}\psi)^2\,dV+\int\psi\nabla^2\psi\,dV$$

Hence $\boxed{\int \psi\vec{\nabla}\psi\cdot d\vec{S} = \int[(\vec{\nabla}\psi)^2+\psi\underbrace{\nabla^2\psi}_{=0\ (31)}]\,d\mathcal{V}}$ (32) GREEN'S IDENTITY

We will return to show that if there are no sources at ∞ then the l.h.s. of (32) vanishes. Accepting this for the moment [See below ✪]

We then have from (32)

$$\int\underbrace{(\vec{\nabla}\psi)^2}_{\text{positive definite (non-negative)}}\,d\mathcal{V}=0 \quad \Rightarrow \quad \boxed{\vec{\nabla}\psi\equiv 0} \tag{33}$$

(34)

But from (30) $\vec{\nabla}\psi\equiv 0 \Rightarrow \vec{W}(x)=\vec{V}_1(x)-V_2(x)\equiv 0 \Rightarrow \boxed{\vec{V}_1(x)=\vec{V}_2(x)}$

In other words, the only way that (28) can hold is if $\vec{V}_1=\vec{V}_2$ so that in the end there is a unique solution.

✪ To complete the proof it remains to show that the l.h.s. of (32) $\to 0$.

Since $\psi(\vec{x})$ is a solution of $\nabla^2\psi(\vec{x})=0$ we ~~can~~ expand $\psi(\vec{x})$ in the form:

$$\psi(\vec{x})\cong R(r)\,Y(\theta,\phi) \tag{35}$$

Then $\nabla^2 \psi(\vec{x}) = 0 \Rightarrow \frac{d}{dr}\left(r^2 \frac{dR}{dr}\right) = 0 \Rightarrow$

$$R(r) = \sum_{n=0}^{\infty} \left(A_n r^n + B_n r^{-n-1}\right) \qquad (36)$$

A_n and B_n are constants chosen to satisfy the boundary conditions appropriate to a given problem. Since we are assuming that there are no sources at ∞ it follows that $A_n \equiv 0$ for all n. Since only the B_n survive the leading term is B_0/r so that [up to a constant]

$$\psi \sim B_0/r \Rightarrow \vec{\nabla}\psi = -B_0 \frac{\hat{r}}{r^2}$$

Hence $\int \psi \vec{\nabla}\psi \cdot d\vec{S} = \int \left(\frac{B_0}{r}\right)\left(-\frac{B_0 \hat{r}}{r^2}\right) \cdot d\vec{S} = -B_0^2 \int \frac{1}{r} \underbrace{\left(\frac{\hat{r} \cdot d\vec{S}}{r^2}\right)}_{d\Omega}$ (37)

$$\boxed{\therefore \int \psi \vec{\nabla}\psi \cdot d\vec{S} \sim -B_0^2 \int \frac{1}{r} d\Omega = -\frac{4\pi}{r} \xrightarrow[r \to \infty]{} 0} \qquad (38)$$

Simply stated, since we assume on physical grounds that there are no sources at ∞, we can find a Gaussian surface for sufficiently large r such that there is no flux of $\psi \vec{\nabla} \psi$ through $d\vec{S}$, and hence the l.h.s of (38) and (32) vanishes. This then completes the proof of uniqueness.

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Side Comment: Returning to (29) and this proof of uniqueness we see that if we have a field $\vec{E}(\vec{x})$ for which

$$\left.\begin{array}{l} \vec{\nabla} \cdot \vec{E}(\vec{x}) = 0 \\ \vec{\nabla} \times \vec{E}(\vec{x}) = 0 \end{array}\right\} \text{ at all points in space} \qquad (39)$$

and if there are no sources at ∞, then $\vec{E}(\vec{x}) \equiv 0$ (40)