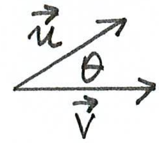
BRIEF REVIEW OF VECTORS

Scalar product: $\vec{u}, \vec{v} \Rightarrow \vec{u}\cdot\vec{V} = |\vec{u}||\vec{V}|\cos\theta$ (1)



Also: We can write

$$\vec{V} = V_1\hat{x} + V_2\hat{y} + V_3\hat{z}$$
$$\vec{u} = u_1\hat{x} + u_2\hat{y} + u_3\hat{z} \qquad (2)$$

Then:

$$\vec{u}\cdot\vec{V} = u_1V_1 + u_2V_2 + u_3V_3 = \sum_i u_iV_i \equiv u_iV_i \qquad (3)$$

Einstein Summation convention

$$(\vec{u}\cdot\vec{V})(\vec{S}\cdot\vec{T}) = u_iV_iS_jT_j \text{ etc.}$$ ← more later

Example of Scalar Product: Energy Conservation

$$\vec{F} = m\vec{a} = m\frac{d\vec{v}}{dt} \Rightarrow \vec{F}\cdot\vec{V} = m\frac{d\vec{V}}{dt}\cdot\vec{v} = m\frac{dV_i}{dt}V_i \qquad (4)$$

$$\therefore \quad \vec{F}\cdot\vec{V} = m\frac{d}{dt}\underbrace{\left(\tfrac{1}{2}V_iV_i\right)}_{\frac{1}{2}v^2} = \frac{d}{dt}\left(\tfrac{1}{2}mv^2\right) \qquad (5)$$

$$\int_a^b \vec{F}\cdot\vec{V}\,dt = \int_a^b dt\left[\frac{d}{dt}\underbrace{\left(\tfrac{1}{2}mv^2\right)}_{KE}\right] = KE(b) - KE(a) \qquad (6)$$

$$\int_a^b F\cdot\frac{d\vec{r}}{dt}\,dt = \int_a^b \vec{F}\cdot d\vec{r} = -(PE(b) - PE(a)) \qquad (7)$$

Hence finally: $\boxed{KE(a) + PE(a) = KE(b) + PE(b)}$ (8)

ENERGY CONSERVATION

Change of Basis & Orthogonal Transformations:

Although we describe a vector by its components relative to some specific coordinate system, <u>the choice of any particular system is arbitrary</u>. This implies that we must understand how vectors appear when described in different coordinate systems.

Consider 2 Cartesian coordinate systems K & K' with the same origin. Then

$$\vec{V} = V_1\hat{x} + V_2\hat{y} + V_3\hat{z} \quad \text{in } K$$
$$\vec{V} = V_1'\hat{x}' + V_2'\hat{y}' + V_3'\hat{z}' \quad \text{in } K' \tag{9}$$

Note: $$\vec{V}\cdot\hat{x} = V_1 \underbrace{\hat{x}\cdot\hat{x}}_{=1} + V_2 \underbrace{\hat{y}\cdot\hat{x}}_{=0} + V_3 \underbrace{\hat{z}\cdot\hat{x}}_{=0} = V_1 \tag{10}$$

Hence we can also write:
$$\vec{V} = (\vec{V}\cdot\hat{x})\hat{x} + (\vec{V}\cdot\hat{y})\hat{y} + (\vec{V}\cdot\hat{z})\hat{z}$$
$$\vec{V} = (\vec{V}\cdot\hat{x}')\hat{x}' + (\vec{V}\cdot\hat{y}')\hat{y}' + (\vec{V}\cdot\hat{z}')\hat{z}' \tag{11}$$

[Later we will introduce <u>DIRAC</u> <u>NOTATION</u> which allows us to write

$$|V\rangle = \sum_n |n\rangle\langle n|V\rangle = \sum_n |n\rangle V_n$$]

Returning to (11), let $\vec{V} = \hat{x}', \hat{y}',$ or $\hat{z}'$. This allows us to relate K & K':

$$\hat{x}' = (\hat{x}'\cdot\hat{x})\hat{x} + (\hat{x}'\cdot\hat{y})\hat{y} + (\hat{x}'\cdot\hat{z})\hat{z}$$
$$\hat{y}' = (\hat{y}'\cdot\hat{x})\hat{x} + (\hat{y}'\cdot\hat{y})\hat{y} + (\hat{y}'\cdot\hat{z})\hat{z} \tag{12}$$
$$\hat{z}' = (\hat{z}'\cdot\hat{x})\hat{x} + (\hat{z}'\cdot\hat{y})\hat{y} + (\hat{z}'\cdot\hat{z})\hat{z}$$

The 9 quantities in (...) are the <u>direction cosines</u> between the various 6 axes. We will show shortly that not all of these 9 quantities are independent. Eq. (11) can be written in matrix notation:

$$\begin{pmatrix} \hat{x}' \\ \hat{y}' \\ \hat{z}' \end{pmatrix} = \begin{pmatrix} \hat{x}'\cdot\hat{x} & \hat{x}'\cdot\hat{y} & \hat{x}'\cdot\hat{z} \\ \hat{y}'\cdot\hat{x} & \hat{y}'\cdot\hat{y} & \hat{y}'\cdot\hat{z} \\ \hat{z}'\cdot\hat{x} & \hat{z}'\cdot\hat{y} & \hat{z}'\cdot\hat{z} \end{pmatrix} \begin{pmatrix} \hat{x} \\ \hat{y} \\ \hat{z} \end{pmatrix} \qquad (13)$$

$$\hookrightarrow R \equiv \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \qquad (14)$$

Briefly:

$$x' = R\,x \quad ; \quad R \equiv (a_{ij})$$

To show that the a_{ij} are not all linearly independent consider

$$\hat{x}'\cdot\hat{x}' = 1 = \underbrace{(\hat{x}'\cdot\hat{x})^2}_{a_{11}^2}\,\underbrace{\hat{x}\cdot\hat{x}}_{1} + \underbrace{(\hat{x}'\cdot\hat{y})^2}_{a_{12}^2}\,\underbrace{\hat{y}\cdot\hat{y}}_{1} + \underbrace{(\hat{x}'\cdot\hat{z})^2}_{a_{13}^2}\,\underbrace{\hat{z}\cdot\hat{z}}_{1} \qquad (15)$$

Hence: $\hat{x}'\cdot\hat{x}' = 1 = a_{11}^2 + a_{12}^2 + a_{13}^2$ (all other terms vanish) (16)

This can be written as: $\sum_{n=1}^{3} a_{1n}^2 = 1$ (17)

More generally (including the other relations of this form)

$$\boxed{\sum_{n=1}^{3} a_{in}^2 = 1 \qquad i = 1, 2, 3} \qquad (18)$$

Similarly, Consider the off-diagonal matrix elements:

$$\hat{x}' \cdot \hat{y}' = 0 = a_{11}a_{21} + a_{12}a_{22} + a_{13}a_{23} = \sum_{n=1}^{3} a_{1n}a_{2n} \qquad (19)$$

There are 3 relations of this form.

All of the relations in (18) & (19) can be subsumed into one equation:

$$\boxed{\sum_{n=1} a_{in}a_{jn} = \delta_{ij} \equiv a_{in}a_{jn}} \qquad (20)$$

Einstein summation convention

$\delta_{ij} = 0 \quad i \neq j$
$\delta_{ij} = 1 \quad i = j$

Eq. (20) comprises 6 equations among the 9 a_{ij}, which leaves only 3 of the a_{ij} as independent.

One can further constrain the transformation matrix $R = (a_{ij})$ by noting that (20) is also satisfied by an <u>inversion</u> described by the matrix

$$P = \begin{pmatrix} -1 & & 0 \\ & -1 & \\ 0 & & -1 \end{pmatrix} \Rightarrow a_{ij} = -\delta_{ij} \qquad (21)$$

$$\hookrightarrow a_{11} = a_{22} = a_{33} = -1 \; ; \; a_{ij} = 0 \quad i \neq j$$

Note that (21) $\Rightarrow \sum_n a_{in}a_{jn} = \sum_n (-\delta_{in})(-\delta_{jn}) = \delta_{ij}$ ✓ (22)

If we want to exclude inversion then we can note that with the additional condition $\det R = +1$ we get what we want:

Hence finally:

$$\boxed{\begin{array}{l} R = (a_{ij}) \text{ where} \\ \delta_{ij} = a_{in}a_{jn} \\ \det R = +1 \end{array}} \qquad (23)$$

Transformation of Arbitrary Vectors Under Rotations

Once the a_{ij} are determined, which fix a specific rotation, we can then determine how any other vector transforms:

$$\vec{V} = V_1'(\hat{x}') + V_2'(\hat{y}') + V_3'(\hat{z}') \tag{24}$$

$$= V_1'(a_{11}\hat{x} + a_{12}\hat{y} + a_{13}\hat{z}) + V_2'(a_{21}\hat{x} + a_{22}\hat{y} + a_{23}\hat{z}) + V_3'(a_{31}\hat{x} + a_{32}\hat{y} + a_{33}\hat{z})$$

$$= \underbrace{(a_{11}V_1' + a_{21}V_2' + a_{31}V_3')}_{\equiv V_1}\hat{x} + \underbrace{(a_{12}V_1' + a_{22}V_2' + a_{32}V_3')}_{\equiv V_2}\hat{y} + \underbrace{(a_{13}V_1' + a_{23}V_2' + a_{33}V_3')}_{\equiv V_3}\hat{z} \tag{25}$$

Hence: $V_1 = a_{11}V_1' + a_{21}V_2' + a_{31}V_3'$; etc.

More generally: $$\boxed{V_i = a_{ni}V_n'} \tag{26}$$

This relation can be inverted by writing

$$\sum_i a_{mi}V_i = \sum_i \sum_n \underbrace{a_{mi}a_{ni}}_{\delta_{mn}} V_n' = V_m' \tag{27}$$

$\longrightarrow$ See comment next page

Hence finally,

$$\boxed{V_m' = a_{mi}V_i} \tag{28}$$

Using (28) we can define a (3-dimensional) vector V_i $(i=1,2,3)$ as an ordered triplet of numbers which transform as in (28) under a rotation defined by (23) above.

Comment on the Orthogonality Relation in (27):

In (23) we write

$$a_{in} a_{jn} \equiv \sum_n a_{in} a_{jn} = \delta_{ij} \qquad (23)$$

If we rename the indices so that $i \to m$ $j \to n$ $n \to i$ then Eq. (23) becomes:

$$a_{mi} a_{ni} = \delta_{mn} \qquad (29)$$

which is the expression used in (27) above

Using the orthogonality relation (29) in this form we can prove that the scalar product of 2 vectors $\vec{u}$ and $\vec{v}$ is invariant under rotations:

$$\vec{u} \cdot \vec{v} = u_i v_i \overset{(26)}{=} (a_{mi} u'_m)(a_{ni} v'_n) = \underbrace{(a_{mi} a_{ni})}_{\delta_{mn}} u'_m v'_n = u'_n v'_n \equiv \vec{u}' \cdot \vec{v}' \checkmark \qquad (30)$$

This is a simple example of the application of a general principle that the laws of physics must be expressed solely in terms of quantities which have well-defined behavior under transformations such as rotations. These include scalars, vectors, and tensors T_{ij} which transform as

$$T'_{mn} = a_{mi} a_{nj} T_{ij}$$

More later when we discuss tensor analysis.

Orthogonality Relations in 2-Dimensions

We illustrate the previous results by explicitly exhibiting the details of the 2-dimensional case: We have

$$R = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \tag{1}$$

$$\text{Orthogonality} \Rightarrow \quad a_{in}a_{jn} = \sum_{n=1}^{2} a_{in}a_{jn} = \delta_{ij} \Rightarrow \tag{2}$$

$$i=1 \;\; j=1 \qquad a_{11}a_{11} + a_{12}a_{12} = a_{11}^2 + a_{12}^2 = \delta_{11} = 1 \tag{3}$$

$$i=1 \;\; j=2 \qquad a_{11}a_{21} + a_{12}a_{22} = \delta_{12} = 0 \tag{4}$$

$$i=2 \;\; j=1 \qquad a_{21}a_{11} + a_{22}a_{12} = \delta_{21} = 0 \tag{5}$$

$$i=2 \;\; j=2 \qquad a_{21}a_{21} + a_{22}a_{22} = \delta_{22} = 1 \tag{6}$$

$$a_{21}^2 + a_{22}^2 = 1$$

We see immediately that Eqs. (3)-(6) are solved by

$$a_{11} = a_{22} = \cos\theta \qquad a_{12} = -a_{21} = \sin\theta \tag{7}$$

$$R = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \Rightarrow \det R = \cos^2\theta + \sin^2\theta = +1 \tag{8}$$

If we did not wish to impose the condition $\det R = +1$, then Eqs. (3)-(6) could also have been solved by

$$R' = \begin{pmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{pmatrix} \tag{9}$$

Vector Product (≡ cross product = outer product)

Given $\vec{u}, \vec{v}$ define $\vec{w} = \vec{u} \times \vec{v}$ where

a) $|\vec{w}| = |\vec{u}||\vec{v}| \sin(\vec{u}, v)$ b) $\hat{w} = \perp$ to $\vec{u}, \vec{v}$ with sense given by r.h. rule (1)

examples: $\hat{x} \times \hat{y} = \hat{z}$ $\vec{L} = \vec{r} \times \vec{p}$ (2)

Equivalently: $\vec{u} \times \vec{v} = \hat{x}(u_2 v_3 - u_3 v_2) + \hat{y}(u_3 v_1 - u_1 v_3) + \hat{z}(u_1 v_2 - u_2 v_1)$

(3)

$$\vec{u} \times \vec{v} = \det \begin{pmatrix} \hat{x} & \hat{y} & \hat{z} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{pmatrix}$$ $u_1 = u_x$, etc.

this relation will be useful later in studying properties of determinants.

To prove various identities involving vector products it is convenient to introduce the symbol (= permutation symbol)

$\boxed{\epsilon_{ijk}}$ $(i, j, k = 1, 2, 3)$

$\epsilon_{123} = \epsilon_{231} = \epsilon_{312} = +1$

$\epsilon_{132} = \epsilon_{321} = \epsilon_{213} = -1$ (4)

all others = 0 (e.g. $\epsilon_{113} = 0$)

Then $(\vec{w})_i \equiv w_i = \epsilon_{ijk} u_j v_k$ (summation understood) (5)

Check: $w_x \equiv w_1 = \epsilon_{1jk} u_j v_k = \underbrace{\epsilon_{123}}_{+1} u_2 v_3 + \underbrace{\epsilon_{132}}_{-1} u_3 v_2 = u_2 v_3 - u_3 v_2$ ✓ (6)

As we will discuss later, ϵ_{ijk} is actually an antisymmetric 3^{rd} rank tensor. We now show that it satisfies the following important identity:

$$\boxed{\epsilon_{ijk}\,\epsilon_{klm} = \delta_{il}\,\delta_{jm} - \delta_{jl}\,\delta_{im}} \quad \text{"}\delta_{il} - \delta_{im}\text{"} \qquad (7)$$

dummy index (can be replaced by any other index not in use)
free index

This identity can be established by choosing, in turn, all possible values for i, j, l, m. However, it can be established more simply by noting that since the only possible values of ϵ_{ijk} are $\pm 1, 0$ the right-hand side (r.h.s.) of (7) must be expressible in terms of δ_{il} ... etc. whose components are also $+1, 0$. Since $\epsilon_{ijk} = -\epsilon_{jik}$ and $\epsilon_{klm} = -\epsilon_{kml}$ the only combination of Kronecker δ-functions which can enter is

$$\epsilon_{ijk}\,\epsilon_{klm} = \underset{\uparrow\,\text{constant}}{C}\left(\delta_{il}\,\delta_{jm} - \delta_{jl}\,\delta_{im}\right) \qquad (8)$$

Since $|\epsilon_{ijk}| = |\epsilon_{klm}| = 1, 0 \Rightarrow |C| = 1 = C = \pm 1$ $(\epsilon_{ijk} = \text{real})$

Finally the sign of C is fixed by taking any one example:

$$\epsilon_{12k}\,\epsilon_{k12} = \sum_k \epsilon_{12k}\,\epsilon_{k12} = \underbrace{\epsilon_{121}\,\epsilon_{112}}_{=0} + \underbrace{\epsilon_{122}\,\epsilon_{212}}_{=0} + \underset{+1}{\epsilon_{123}}\,\underset{+1}{\epsilon_{312}} = +1 \qquad (9)$$

$$= C(\delta_{11}\delta_{22} - \delta_{21}\,\delta_{12}) = C \Rightarrow C = +1 \checkmark \qquad (10)$$

From the basic identity in (7) we have (setting $l=j$)

$$\epsilon_{ijk}\,\epsilon_{kjm} = \underbrace{\delta_{ij}\,\delta_{jm}}_{\delta_{im}} - \delta_{jj}\,\delta_{im} \tag{11}$$

We note that $\boxed{\delta_{jj} \equiv \sum_j \delta_{jj} = \delta_{11} + \delta_{22} + \delta_{33} = 3}$. Hence (12)

$$\boxed{\epsilon_{ijk}\,\epsilon_{kjm} = \delta_{im} - 3\delta_{im} = -2\,\delta_{im}} \tag{13}$$

Finally, setting $m=i$ in (13) gives $\boxed{\epsilon_{ijk}\,\epsilon_{kji} = -2\delta_{ii} = -6}$ (14)

Collecting these results together, and rearranging some indices, we get

$$\boxed{\begin{aligned} \epsilon_{ijk}\,\epsilon_{klm} &= \delta_{il}\,\delta_{jm} - \delta_{jl}\,\delta_{im} \\ \epsilon_{ijk}\,\epsilon_{kmj} &= 2\,\delta_{im} \\ \epsilon_{ijk}\,\epsilon_{ijk} &= 6 \end{aligned}} \tag{15}$$

Applications:

[1] $\vec{A}\cdot(\vec{B}\times\vec{C}) = A_i\,(\vec{B}\times\vec{C})_i \overset{(5)}{=} A_i\,(\epsilon_{ijk}\,B_j\,C_k) = \epsilon_{ijk}\,A_i\,B_j\,C_k$

$$= \det\begin{pmatrix} A_1 & A_2 & A_3 \\ B_1 & B_2 & B_3 \\ C_1 & C_2 & C_3 \end{pmatrix} \tag{16}$$

Using the antisymmetry of ϵ_{ijk} we can write:

$$\vec{A}\cdot\vec{B}\times\vec{C} = \epsilon_{ijk}\,A_i\,B_j\,C_k = +\epsilon_{kij}\,C_k\,A_i\,B_j = \vec{C}\cdot\vec{A}\times\vec{B} \tag{17}$$

$$\therefore\quad \vec{A}\cdot(\vec{B}\times\vec{C}) = \vec{C}\cdot(\vec{A}\times\vec{B}) \tag{18}$$

$$\text{also} \quad = \vec{B}\cdot\vec{C}\times\vec{A}$$

[2] Consider next simplifying $\vec{A}\times(\vec{B}\times\vec{C})$:

$$[\vec{A}\times(\vec{B}\times\vec{C})]_i = \epsilon_{ijk}\, A_j\, \underbrace{(\vec{B}\times\vec{C})_k}_{\epsilon_{k\ell m} B_\ell C_m} \qquad (19)$$

Hence $[\vec{A}\times(\vec{B}\times\vec{C})]_i = \epsilon_{ijk}\,\epsilon_{k\ell m}\, A_j B_\ell C_m = (\delta_{i\ell}\delta_{jm} - \delta_{\ell j}\delta_{im}) A_j B_\ell C_m$ (20)

$$= B_i(\vec{A}\cdot\vec{C}) - C_i(\vec{A}\cdot\vec{B}) \Leftrightarrow \boxed{\vec{A}\times(\vec{B}\times\vec{C}) = \vec{B}(\vec{A}\cdot\vec{C}) - \vec{C}(\vec{A}\cdot\vec{B})} \qquad (21)$$

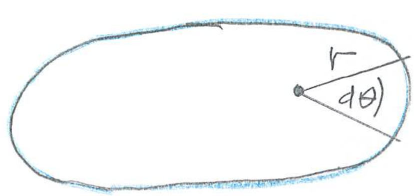
"BAC" – "CAB"

[3] Differentiation of Cross Products

$$\vec{L} = \vec{r}\times\vec{p} \Rightarrow \frac{d\vec{L}}{dt} = \underbrace{\frac{d\vec{r}}{dt}\times\vec{p}}_{\vec{v}\times m\vec{v}=0} + \vec{r}\times\frac{d\vec{p}}{dt} = \vec{r}\times\vec{F} \qquad (22)$$

For a central force $\vec{r}\,\|\,\vec{F} \Rightarrow d\vec{L}/dt = 0$ (23)

This is Kepler's 2ND Law, usually stated as the constancy of the areal velocity):



$$dA = \tfrac{1}{2}(r\,d\theta)r = \tfrac{1}{2}r^2 d\theta \qquad (24)$$

$$\frac{dA}{dt} = \frac{1}{2}r^2\frac{d\theta}{dt} = \frac{1}{2}r^2\omega \qquad (25)$$

$$L = mvr = m\omega r^2 \Rightarrow \frac{dA}{dt} = \frac{1}{2}r^2\omega = \frac{L}{2m} = \text{constant} \qquad (26)$$

Hence the constancy of the areal velocity is a consequence of the constancy of the (orbital) angular-momentum L.

Scalar, Vector & Tensor Fields

[1] A <u>scalar</u> field is a function $\phi(\vec{r})$ which assigns to each point $\vec{r}$ in space a scalar ϕ. [Examples: temperature distribution, density distribution, electromagnetic scalar potential.]

[2] A <u>vector</u> field $\vec{A}(\vec{r})$ assigns to each point a vector $\vec{A}$. [Examples: electromagnetic fields $\vec{E}(\vec{r})$, $\vec{B}(\vec{r})$; gravitational field $\vec{g}(\vec{r})$; velocity field in a fluid $\vec{v}(\vec{r})$]

[3] A <u>tensor</u> field $g_{ij}(\vec{r})$ assigns to each point in space a tensor (g_{ij}) quantity. [Examples: metric tensor g_{ij}, energy-momentum tensor T_{ij}, electromagnetic field tensor $F_{\mu\nu}$]

The familiar differential operators act on these fields and produce other fields:

$$\text{input field} \longrightarrow \boxed{\begin{array}{l}\text{Gradient, curl, divergence}\\ \text{Laplacian}\end{array}} \xrightarrow{\text{output field}}$$

PHYSICAL INTERPRETATION OF DIV, GRAD, CURL, ...

See: H. M. Schey, Div, Grad, Curl, and All That (Norton, New York, 1973)

Gradient: Given a scalar function $u = u(\vec{r}) = u(x,y,z)$ we define

$$\text{grad}\, u \equiv \vec{\nabla} u = \hat{x}\frac{\partial u}{\partial x} + \hat{y}\frac{\partial u}{\partial y} + \hat{z}\frac{\partial u}{\partial z} = \text{VECTOR} \qquad (1)$$

One can develop a physical picture of $\vec{\nabla} u$ can be obtained by noting that the scalar change du of u is given by

$$du = \frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy + \frac{\partial u}{\partial z}dz = \underbrace{\left(\hat{x}\frac{\partial u}{\partial x} + \hat{y}\frac{\partial u}{\partial y} + \hat{z}\frac{\partial u}{\partial z}\right)}_{\vec{\nabla} u} \cdot \underbrace{\left(\hat{x}\,dx + \hat{y}\,dy + \hat{z}\,dz\right)}_{d\vec{r}} \qquad (2)$$

Hence

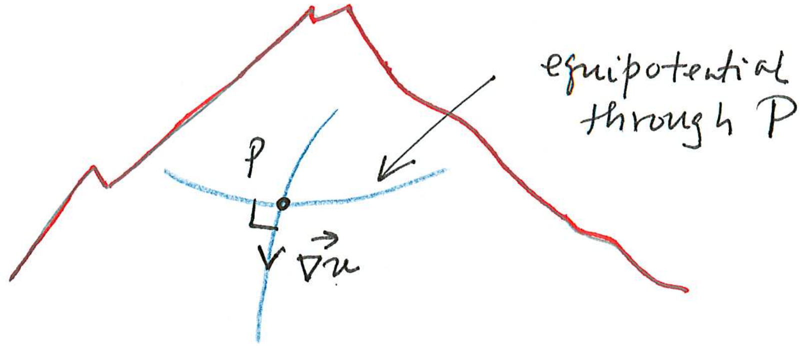
$$\boxed{du = \vec{\nabla} u \cdot d\vec{r} = |\vec{\nabla} u|\,|d\vec{r}|\cos\theta} \qquad (3)$$

$\cos\theta$ is the angle between $\vec{\nabla} u$ and $d\vec{r}$. We see that du is a maximum when $\vec{\nabla} u$ and $d\vec{r}$ point in the same direction, so that

$$|\vec{\nabla} u| = |du| / |d\vec{r}| \qquad (4)$$

Summary: If we move from $\vec{r}$ to $(\vec{r}+dr)$, then $u(\vec{r})$ will change by an amount du. From (3) we see that this change will be a maximum when $d\vec{r}$ is chosen to be in the direction of $\vec{\nabla} u$. So $\vec{\nabla} u$ points in the direction in which $u(\vec{r})$ increases most rapidly. Hence $\vec{\nabla} u$ extracts from $u(\vec{r})$ the information about the direction in which $u(\vec{r})$ is changing most rapidly.

Example:



Consider a mountain where we characterize each point by a gravitational potential energy $u(P = x, y, z)$. Then $\vec{\nabla} u(P)$ points along the fall line which is the path of steepest descent. If a ski came loose, this is the path it would take.

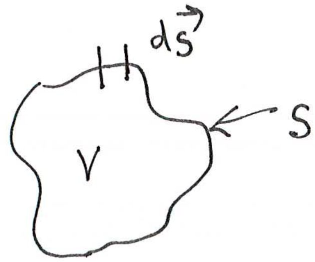
Divergence: This acts on vector fields to produce a scalar

$$\operatorname{div}\vec{A} \equiv \vec{\nabla}\cdot\vec{A} \equiv \vec{\partial}\cdot\vec{A} = \left(\hat{x}\frac{\partial}{\partial x} + \hat{y}\frac{\partial}{\partial y} + \hat{z}\frac{\partial}{\partial z}\right)\cdot\left(\hat{x}A_x + \hat{y}A_y + \hat{z}A_z\right)$$

$$= \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z} \equiv \partial_i A_i = \underline{\text{SCALAR}}$$

One important identity involving $\vec{\nabla}\cdot\vec{A}$ is GAUSS' THEOREM:

$$\int_V \vec{\nabla}\cdot\vec{A}\, dV = \int_S \hat{n}\cdot\vec{A}\, dS = \int_S \vec{A}\cdot(\hat{n}\,dS) = \int_S \vec{A}\cdot d\vec{S}$$



CURL : This operator acts on vector fields and produces another vector field.

Define $\partial_x \equiv \partial/\partial x$ etc.

$$\text{curl}\,\vec{A} = \vec{\nabla}\times\vec{A} = \vec{\partial}\times\vec{A} = \hat{x}(\partial_y A_z - \partial_z A_y) + \hat{y}(\partial_z A_x - \partial_x A_z) + \hat{z}(\partial_x A_y - \partial_y A_x) \quad (1)$$

$$= \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \partial_x & \partial_y & \partial_z \\ A_x & A_y & A_z \end{vmatrix} \Rightarrow \boxed{(\vec{\nabla}\times\vec{A})_i = \sum_{j,k} \epsilon_{ijk}\,\partial_j A_k} \quad (2)$$

determinant

The last representation is useful in proving certain identities such as:

$\vec{\partial} \equiv \vec{\nabla}$

$$\vec{\nabla}\cdot(\vec{\nabla}\times\vec{A}) = \partial_i(\vec{\nabla}\times\vec{A})_i = \partial_i(\epsilon_{ijk}\partial_j A_k) \quad (3)$$

$$= \underbrace{\epsilon_{ijk}}_{\text{antisymmetric in } (i\Leftrightarrow j)}\ \underbrace{\partial_i\partial_j}_{\text{symmetric in } (i\Leftrightarrow j)} A_k \equiv 0 \quad (4)\ (5)$$

To show this is zero: $\epsilon_{ijk}\partial_i\partial_j \overset{(i\Leftrightarrow j)}{=} +\epsilon_{jik}\partial_j\partial_i = -\epsilon_{ijk}\partial_j\partial_i$ (6)

$= -\epsilon_{ijk}\partial_i\partial_j = 0$

Alternatively, do this by components: $\epsilon_{ijk}\partial_i\partial_j \to \epsilon_{ij3}\partial_i\partial_j$ (7)

$= \epsilon_{123}\partial_1\partial_2 + \epsilon_{213}\partial_2\partial_1 = +\partial_1\partial_2 - \partial_2\partial_1 \equiv 0$ etc.

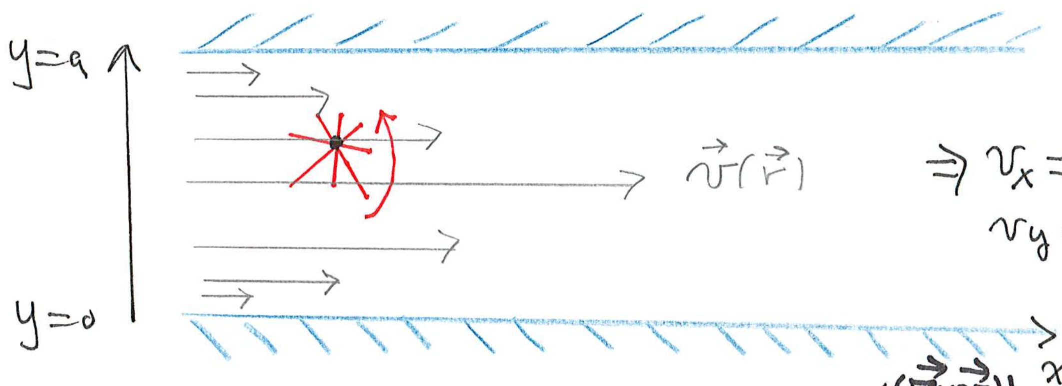
Hence returning to (3) we have shown that $\boxed{\vec{\nabla}\cdot(\vec{\nabla}\times\vec{A}) \equiv 0}$ (8)

It follows that any field (such as the magnetic field $\vec{B}$) which can be expressed as the curl of another field ($\vec{B} = \vec{\nabla}\times\vec{A}$) has zero divergence.

Physical Interpretation of Curl:

Given a vector field [such as the velocity $\vec{v}(\vec{r})$ in the example below], that field will have a non-vanishing curl at $\vec{r}$ if a minute "paddle-wheel" placed at $\vec{r}$ will rotate. This can happen even if all the field lines for $\vec{v}(\vec{r})$ are straight:

As an example consider the flow of a river:



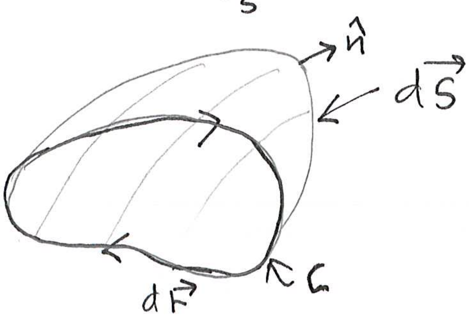
$$\Rightarrow v_x = v_0\, y(y-a)$$
$$v_y = 0$$

In this example $dv_x/dy = (2y-a)\,v_0 \Rightarrow |(\vec{\nabla}\times\vec{v})|$ is a ~~maximum~~ minimum at $a/2$

It then follows that $\boxed{(\vec{\nabla}\times\vec{v})_z = \hat{z}\,(\partial_x v_y - \partial_y v_x) = -\hat{z}\,v_0\,(2y-a) \neq 0.}$

Hence there is in general a non-zero curl, even though the field lines are straight. Note that $(\vec{\nabla}\times\vec{v})_z = 0$ at $y = a/2$ as expected on symmetry grounds.

STOKES' THEOREM: $$\int_S (\vec{\nabla}\times\vec{A})\cdot\hat{n}\,dS = \int_S (\vec{\nabla}\times\vec{A})\cdot d\vec{S} = \oint_C \vec{A}\cdot d\vec{r}$$

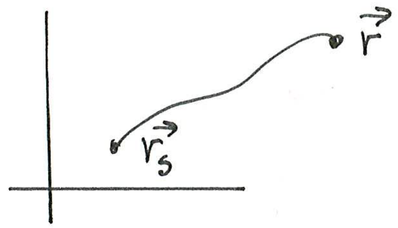


CONSERVATIVE FIELDS :

One can gain additional insight into the meaning of the curl by noting that the condition for the existence of a CONSERVATIVE FIELD $\vec{F}(\vec{r})$ is

$$\boxed{\vec{\nabla}\times\vec{F}(\vec{r})=0} \qquad (1)$$

Proof:



We define the work W done by a force $\vec{F}(\vec{r})$ along any path as

$$W=\int_{\vec{r}_1}^{\vec{r}_2}\vec{F}\cdot d\vec{r} \qquad (2)$$

The potential $V(\vec{r})$ can then be defined as $\boxed{V(\vec{r})=-\int_{\vec{r}_s}^{\vec{r}}\vec{F}(\vec{r}\,')\cdot d\vec{r}\,'}$ (3)
where $\vec{r}_s$ is some standard reference point.

Then (3) $\Rightarrow$ $\boxed{dV(\vec{r}) = -\vec{F}(\vec{r})\cdot d\vec{r}}$ (4)

But this is equivalent to writing $\boxed{\vec{F}(r)=-\vec{\nabla}V(\vec{r})}$ (5) Since (5) $\Rightarrow$

$$\vec{F}\cdot d\vec{r}=-\vec{\nabla}V\cdot d\vec{r}=-\left(\hat{x}\frac{\partial V}{\partial x}+\hat{y}\frac{\partial V}{\partial y}+\hat{z}\frac{\partial V}{\partial z}\right)\cdot\left(\hat{x}dx+\hat{y}dy+\hat{z}dz\right)$$

$$=-\left(\frac{\partial V}{\partial x}dx+\frac{\partial V}{\partial y}dy+\frac{\partial V}{\partial z}dz\right)\equiv -dV \qquad (6)$$

Hence altogether:

$$\boxed{V(r)=-\int_{r_s}^{r}\vec{F}(\vec{r}\,')\cdot d\vec{r}\,' \Leftrightarrow dV(\vec{r})=-\vec{F}\cdot d\vec{r} \Leftrightarrow \vec{F}(\vec{r})=-\vec{\nabla}V(\vec{r})} \qquad (7)$$

Suppose now that we have a field $\vec{F}(\vec{r})$ that can be represented as the gradient of some potential $V(\vec{r})$ as in (7). It is straightforward to show that such a field $\vec{F}$ satisfies

$$\vec{\nabla}\times\vec{F}=0$$

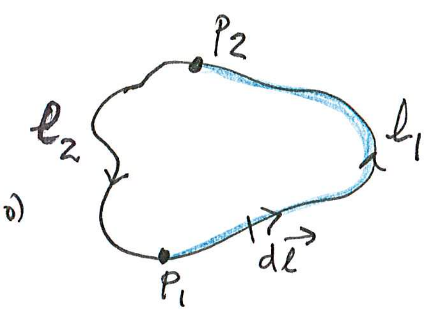
$$(\vec{\nabla}\times\vec{F})_i = \epsilon_{ijk}\,\partial_j F_k \qquad \uparrow = -\partial_k V \quad \Big\} \text{ This is the same as } \vec{F}=-\vec{\nabla}V$$

$$(\vec{\nabla}\times\vec{F})_i = -\underbrace{\epsilon_{ijk}\,\partial_j\partial_k}_{=0} V \equiv 0 \qquad \leftarrow \text{using } 17(4) \tag{8}$$

Hence $\boxed{\vec{F}=-\vec{\nabla}V \Rightarrow \vec{\nabla}\times\vec{F}=0}$ (9)

We next show that the implication goes the other way too:

Using Stokes' theorem [p.18]



$$\underbrace{\int_S \vec{\nabla}\times\vec{F}\cdot d\vec{S}}_{0 \text{ from } (9)} = \oint_C \vec{F}\cdot d\vec{l} = 0 \tag{10}$$

Then
$$0=\oint \vec{F}\cdot d\vec{l} = \underbrace{\int_{P_1}^{P_2} \vec{F}\cdot d\vec{l}}_{\text{along } l_1} + \underbrace{\int_{P_2}^{P_1} \vec{F}\cdot d\vec{l}}_{\text{along } l_2} \tag{11}$$

From (11)
$$\underbrace{\int_{P_1}^{P_2} \vec{F}\cdot d\vec{l}}_{\text{over } l_1} = -\underbrace{\int_{P_2}^{P_1} \vec{F}\cdot d\vec{l}}_{\text{over } l_2} = +\underbrace{\int_{P_1}^{P_2} \vec{F}\cdot d\vec{l}}_{\text{over } l_2} \tag{12}$$

Hence when $\vec{\nabla}\times\vec{F}=0$ the value of $\int \vec{F}\cdot d\vec{l}$ between any 2 points P_1 and P_2 is <u>independent of the path</u> (l_1 or l_2) between these points. But this $\Rightarrow$ $\boxed{\vec{F}\cdot d\vec{l} = -dV}$: (13)

We note that if $\vec{F}\cdot d\vec{l} = -dV$ then

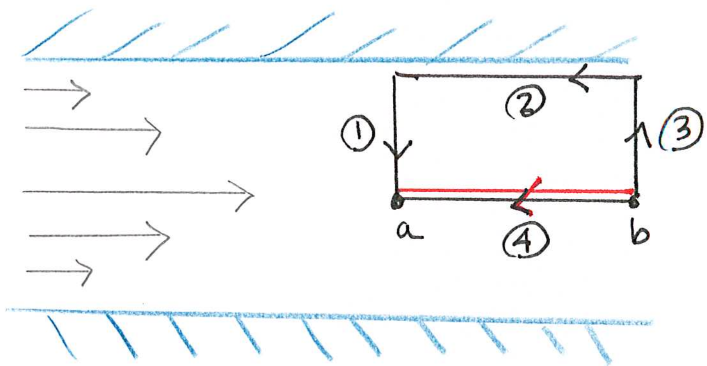
$$\int_{P_1}^{P_2} \vec{F}\cdot d\vec{l} = -\int_{P_1}^{P_2} dV = V(P_1) - V(P_2) \leftarrow \underline{\text{independent of the path}} \quad (14)$$

We have shown on p.18.1 that $dV = \vec{\nabla}V\cdot d\vec{l}$, and hence having shown that $\vec{\nabla}\times\vec{F} = 0 \Rightarrow \vec{F}\cdot d\vec{l} = -dV$ we have also shown that

$$\vec{F}\cdot d\vec{l} = -dV = -\vec{\nabla}V\cdot d\vec{l} \Rightarrow \boxed{\vec{F} = -\vec{\nabla}V} \quad (15)$$

Hence altogether: $$\boxed{\vec{F} = -\vec{\nabla}V \Leftrightarrow \vec{\nabla}\times\vec{F} = 0} \quad (16)$$

Physical Picture: Using the river example from p.18 we can see physically why a velocity field for which $\vec{\nabla}\times\vec{v} \neq 0$ is not conservative:



Clearly the work done in going from $b \to a$ along ④ is greater than along the path ③②①

The Laplacian: $\nabla^2 \equiv \Delta$ [also $\boxed{4}$, ⑤, ...]

This is an important operator which arises in many interesting physics equations. The Laplacian of a scalar field is defined by

$$\nabla^2 u(\vec{r}) = \vec{\nabla}\cdot(\vec{\nabla}u(\vec{r})) = \left(\hat{x}\frac{\partial}{\partial x} + \hat{y}\frac{\partial}{\partial y} + \hat{z}\frac{\partial}{\partial z}\right)\cdot\left(\hat{x}\frac{\partial u}{\partial x} + \hat{y}\frac{\partial u}{\partial y} + \hat{z}\frac{\partial u}{\partial z}\right) \tag{1}$$

$$= \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}$$

Notes: a) We will later define the Laplacian of a vector field, $\nabla^2\vec{V}$.

b) In the upcoming discussion of tensor analysis we will show how to express ∇^2 in an arbitrary coordinate system.

Physical Interpretation of the Laplacian:

Let $u(\vec{r})$ have the value u_0 at the origin of a Cartesian coordinate system. Construct a cube of side a around the origin. The average value $\bar{u}$ of u inside this cube is then given by

$$\bar{u} = \frac{1}{a^3}\int_{-a/2}^{a/2} dx \int_{-a/2}^{a/2} dy \int_{-a/2}^{a/2} dz \; u(x,y,z) \tag{2}$$

Assuming that a is small, so that $u(x,y,z)$ varies slowly over the cube, expand $u(x,y,z)$ in a Taylor series about the origin:

$$u(x,y,z) = u_0 + x\left(\frac{\partial u}{\partial x}\right)_0 + y\left(\frac{\partial u}{\partial y}\right)_0 + z\left(\frac{\partial u}{\partial z}\right)_0 \tag{3}$$

$$+ \frac{1}{2}x^2\left(\frac{\partial^2 u}{\partial x^2}\right)_0 + \frac{1}{2}y^2\left(\frac{\partial^2 u}{\partial y^2}\right)_0 + \frac{1}{2}z^2\left(\frac{\partial^2 u}{\partial z^2}\right)_0 + xy\left(\frac{\partial^2 u}{\partial x\partial y}\right)_0 + xz\left(\frac{\partial^2 u}{\partial x\partial z}\right)_0 + yz\left(\frac{\partial^2 u}{\partial y\partial z}\right)_0 + \cdots$$

Inserting the expansion (3) into (2) we see that the terms linear in x or y or z vanish since

$$\int_{-a/2}^{a/2} dx \cdot x \cdots = \frac{1}{2} x^2 \Big|_{-a/2}^{a/2} = 0 \quad \text{etc.} \tag{4}$$

The same holds for the terms proportional to xy, xz, and yz. Thus the only surviving contributions are of the form:

$$\left(\frac{\partial^2 u}{\partial x^2}\right)_0 \int_{-a/2}^{a/2} dx \left(\tfrac{1}{2}x^2\right) \int_{-a/2}^{a/2} dy \int_{-a/2}^{a/2} dz = \left(\frac{\partial^2 u}{\partial x^2}\right)_0 \left[\frac{1}{2}\cdot\frac{1}{3}x^3\right]_{-a/2}^{a/2} \left[y\right]_{-a/2}^{a/2} \left[z\right]_{-a/2}^{a/2} = \frac{a^5}{24}\left(\frac{\partial^2 u}{\partial x^2}\right)_0 \tag{5}$$

Combining Eqs. (2) & (5) we have

$$\bar{u} = u_0 + \frac{1}{a^3}\cdot\frac{a^5}{24}\left[\left(\frac{\partial^2 u}{\partial x^2}\right)_0 + \left(\frac{\partial^2 u}{\partial y^2}\right)_0 + \left(\frac{\partial^2 u}{\partial z^2}\right)_0\right] = u_0 + \frac{a^2}{24}\left(\nabla^2 u\right)_0 \tag{6}$$

Hence finally:

$$\boxed{\nabla^2 u(x=y=z=0) = \frac{24}{a^2}\left[\bar{u} - u_0\right]} \tag{7}$$

Thus the Laplacian measures the difference between the value of a function at a given point, u_0, and the average of the function in an infinitesimal neighborhood around the point. This gives a physical interpretation of the heat/diffusion equations:

$$\nabla^2 u(\vec{r},t) = K \frac{\partial u(\vec{r},t)}{\partial t} \Bigg\}$$

u = temperature; K^{-1} = heat conductivity

u = density; K^{-1} = diffusion constant

<u>In 4-dimensions:</u>

$$\Box u(\vec{x},t) = \left(\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right) u(\vec{x},t) = 0$$

One of the most important relations involving ∇^2 is:

$$\boxed{\nabla^2\left(\frac{1}{r}\right) = -4\pi\,\delta^3(\vec{r})} \quad (1) \qquad \delta^3(r) \equiv \delta(x)\,\delta(y)\,\delta(z)$$

This has important experimental and theoretical consequences which we discuss below. To establish this result we first show that $\nabla^2(1/r) = 0$ when $r \neq 0$: Using $r = (x^2+y^2+z^2)^{1/2}$

$$\nabla^2(1/r) = \vec{\nabla}\cdot\vec{\nabla}(1/r) = \vec{\nabla}\cdot\left[\hat{x}\frac{\partial}{\partial x}(1/r) + \hat{y}\frac{\partial}{\partial y}(1/r) + \hat{z}\frac{\partial}{\partial z}(1/r)\right] \quad (2)$$

$$\frac{\partial}{\partial x}(1/r) = -x(x^2+y^2+z^2)^{-3/2} = \frac{-x}{r^3} \quad (3)$$

Hence $$\frac{\partial^2}{\partial x^2}\left(\frac{1}{r}\right) = -\frac{1}{r^3} + \frac{3x^2}{r^5} \quad (4)$$

Hence $$\nabla^2\left(\frac{1}{r}\right) = \vec{\nabla}\cdot\vec{\nabla}\left(\frac{1}{r}\right) = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right)\left(\frac{1}{r}\right) \quad (5)$$

$$= \left(-\frac{1}{r^3} + \frac{3x^2}{r^5}\right) + \left(-\frac{1}{r^3} + \frac{3y^2}{r^5}\right) + \left(-\frac{1}{r^3} + \frac{3z^2}{r^5}\right) \quad (6)$$

$$\therefore \nabla^2\left(\frac{1}{r}\right) = -\frac{3}{r^3} + \frac{3}{r^5}(x^2+y^2+z^2) \equiv 0 \quad (r \neq 0) \quad (7)$$

When $r=0$ the various differentiations leading to (7) would no longer be valid, so we must be more careful: Consider

$$\int \nabla^2\left(\frac{1}{r}\right) d^3r \equiv \int \vec{\nabla}\cdot\vec{\nabla}\left(\frac{1}{r}\right) dV = \int \vec{\nabla}\left(\frac{1}{r}\right)\cdot d\vec{S} \quad (8)$$

Gauss' Theorem

$$\vec{\nabla}\left(\frac{1}{r}\right) = (\hat{\imath}\,\partial_x + \hat{\jmath}\,\partial_y + \hat{k}\,\partial_z)\left(\frac{1}{r}\right) = -(\hat{\imath}x + \hat{\jmath}y + \hat{k}z)\frac{1}{r^3} = -\frac{\vec{r}}{r^3} \quad (9)$$

$$\therefore \int \nabla^2\left(\frac{1}{r}\right) dV = -\int \frac{\vec{r}\cdot d\vec{S}}{r^3} \equiv -\int d\Omega = -4\pi \quad (10)$$

(We note from (10) that since $\vec{F}\cdot d\vec{s}$ has the same sign at all points on the surface of a sphere that $\int \vec{r}\cdot d\vec{s}/r^3$ cannot be zero when integrated over the surface.)

It follows from (10) that $\underline{\nabla^2(1/r) \text{ cannot be zero everywhere}}$, or else Eq. (10) could not hold. In fact we see that (7) & (10) can be made compatible if (1) holds since then

$$\int \nabla^2\left(\frac{1}{r}\right) dV = \int \nabla^2\left(\frac{1}{r}\right) dx\,dy\,dz = -4\pi \int \delta(x)\delta(y)\delta(z)\,dx\,dy\,dz$$

$$= -4\pi \int \delta(x)dx \int \delta(y)dy \int \delta(z)\,dz = -4\pi V \qquad (11)$$

Experimental Consequences:

For Newtonian gravity the potential energy is

$$V_{12}^{(N)}(r) = -\frac{G\,m_1 m_2}{r} \qquad (12)$$

and for the the Coulomb interaction $V_{12}^{(C)}(r) = \dfrac{+q_1 q_2}{r}$ (13)

In both cases $\nabla^2 V_{12}^{(N,C)} = 0 \quad (r \neq 0)$. (14)

However, suppose that in the gravitational case there was an additional contribution such that

$$V_{12}(r) = -\frac{G m_1 m_2}{r}\left(1 + \alpha e^{-r/\lambda}\right) \quad ; \ \alpha, \lambda = \text{constants} \qquad (15)$$

Then by an elementary calculation we find

$$\boxed{\nabla^2 V_{12}(r) = \frac{\alpha G m_1 m_2}{\lambda^2 r}\, e^{-r/\lambda} \neq 0} \qquad (16)$$

Measuring $\nabla^2 V_{12}(r) \neq 0$ is thus a means of testing for deviations from Newton's law of gravity, or from Coulomb's law.

CARTESIAN TENSOR NOTATION FOR VECTOR IDENTITIES

Definitions:

1) $\vec{u}\cdot\vec{V} = \sum_i u_i V_i \equiv u_i V_i$

2) $(\vec{u}\times\vec{V})_i = \sum_{j,k} \epsilon_{ijk} u_j V_k \equiv \epsilon_{ijk} u_j V_k$

3) $(\vec{\nabla}\phi)_i = \partial_i \phi$ $\qquad \vec{\nabla} \equiv \vec{\partial}$

4) $\vec{\nabla}\cdot\vec{V} = \partial_i V_i$ $\qquad \nabla_i \equiv \partial_i$

5) $(\vec{\nabla}\times\vec{V})_i = \epsilon_{ijk}\partial_j V_k$

6) $\nabla^2\phi = \vec{\nabla}\cdot(\vec{\nabla}\phi) = \partial_i(\partial_i\phi) = \partial_i\partial_i\phi$

7) $(\nabla^2\vec{V})_i = \underbrace{\partial_j\partial_j}_{\text{dummy indices}} \underbrace{V_i}_{\text{free index}}$

Some Simple Theorems:

a) $\vec{\nabla}\times(\vec{\nabla}\phi) = 0$ $\qquad (8)$

$\hookrightarrow [\vec{\nabla}\times(\vec{\nabla}\phi)]_i = \epsilon_{ijk}\partial_j(\partial_k\phi) = \epsilon_{ijk}\partial_j\partial_k\phi = 0$ $\qquad (9)$

b) $\vec{\nabla}\cdot(\vec{\nabla}\times\vec{V}) = 0$; see p.17 $\qquad (10)$

c) $\vec{\nabla}\cdot(\vec{u}\times\vec{V}) = \partial_i(\epsilon_{ijk}u_j V_k) = \epsilon_{ijk}(u_j\partial_i V_k + V_k\partial_i u_j)$

$= -\epsilon_{jik}u_j\partial_i V_k + \epsilon_{kij}V_k\partial_i u_j = -\vec{u}\cdot(\vec{\nabla}\times\vec{V}) + \vec{V}\cdot(\vec{\nabla}\times\vec{u})$ $\qquad (11)$

$$\therefore \boxed{\vec{\nabla}\cdot(\vec{u}\times\vec{V}) = \vec{V}\cdot(\vec{\nabla}\times\vec{u}) - \vec{u}\cdot(\vec{\nabla}\times\vec{V})} \qquad (12)$$

d) Laplacian of a Vector Field

Consider $\vec{\nabla}\times(\vec{\nabla}\times\vec{V})$:

$$[\vec{\nabla}\times(\vec{\nabla}\times\vec{V})]_i = \epsilon_{ijk}\,\partial_j\,(\epsilon_{klm}\,\partial_l V_m) = \epsilon_{ijk}\,\epsilon_{klm}\,\partial_j\partial_l\,V_m \qquad (13)$$

$$= (\delta_{il}\,\delta_{jm} - \delta_{jl}\,\delta_{im})\,\partial_j\partial_l V_m = \partial_i(\partial_j V_j) - (\partial_j\partial_j)V_i = \partial_i(\vec{\nabla}\cdot\vec{V}) - \nabla^2 V_i \qquad (14)$$

$$\therefore\ [\vec{\nabla}\times(\vec{\nabla}\times\vec{V})]_i = -\nabla^2 V_i + \partial_i\,(\vec{\nabla}\cdot\vec{V}) \qquad (15)$$

In more familiar notation:

$$\vec{\nabla}\times(\vec{\nabla}\times\vec{V}) = -\nabla^2\vec{V} + \vec{\nabla}(\vec{\nabla}\cdot\vec{V}) \qquad (16)$$

Hence finally: $$\boxed{\nabla^2\vec{V} = \vec{\nabla}(\vec{\nabla}\cdot\vec{V}) - \vec{\nabla}\times(\vec{\nabla}\times\vec{V})} \qquad (17)$$

Since all the quantities on the r.h.s. are well-defined this defines what we mean by $\nabla^2\vec{V}$.