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PHYS 600 – HW7 SOLUTIONS

PROBLEM 1 if our set of functions $\{ f_i(x) \}$ having the domain $[a, b]$ is complete in the sense of uniform convergence then any function g can be expanded as

$$g(x) = \sum_{i=1}^{\infty} a_i f_i(x) \quad \text{or , using the notation } g_n(x) \equiv \sum_{i=1}^{n} a_i f_i(x) \quad , \quad \lim_{n\to\infty} \sup_{x\in[a,b]} \left| g(x) - g_n(x) \right| = 0$$

$$\left| \int_a^b (g(x) - g_n(x))^2 \, dx \right| \le \left(\sup_{x\in[a,b]} \left| g(x) - g_n(x) \right| \right)^2 (b-a) \quad \text{hence} \quad \lim_{n\to\infty} \left| \int_a^b (g(x) - g_n(x))^2 \, dx \right| = 0$$

therefore the set $\{ f_i(x) \}$ is complete in the sense of convergence in the mean

PROBLEM 2 apply the Gram – Schmidt orthogonalization procedure

$$\overline{P}_0(x) = \alpha \text{ and } 1 = \int_{-1}^{1} (\overline{P}_0(x))^2 \, dx = 2\alpha^2 \text{ therefore } \alpha = \frac{1}{\sqrt{2}} \text{ and } \overline{P}_0(x) = \frac{1}{\sqrt{2}}$$

$$\overline{P}_1(x) = \beta_1 \overline{P}_0(x) + \beta_2 x \,,\; 0 = \int_{-1}^{1} \overline{P}_0(x) \overline{P}_1(x) \, dx = \int_{-1}^{1} \overline{P}_0(x) (\beta_1 \overline{P}_0(x) + \beta_2 x) \, dx = \beta_1 + \beta_2 \int_{-1}^{1} \overline{P}_0(x) \, x \, dx = \beta_1$$

$$\text{and } 1 = \int_{-1}^{1} (\overline{P}_1(x))^2 \, dx = \beta_2^2 \int_{-1}^{1} x^2 \, dx = \frac{2}{3} \beta_2^2 \text{ therefore } \beta_1 = 0 \,,\; \beta_2 = \sqrt{\frac{3}{2}} \text{ and } \overline{P}_1(x) = \sqrt{\frac{3}{2}} \, x$$

$$\overline{P}_2(x) = \gamma_1 \overline{P}_0(x) + \gamma_2 \overline{P}_1(x) + \gamma_3 x^2, \; 0 = \int_{-1}^{1} \overline{P}_0(x) \overline{P}_2(x) \, dx = \int_{-1}^{1} \overline{P}_0(x) (\gamma_1 \overline{P}_0(x) + \gamma_2 \overline{P}_1(x) + \gamma_3 x^2) \, dx = \gamma_1 + \frac{\gamma_3}{\sqrt{2}} \frac{2}{3},$$

$$0 = \int_{-1}^{1} \overline{P}_1(x) \overline{P}_2(x) \, dx = \int_{-1}^{1} \overline{P}_1(x) (\gamma_1 \overline{P}_0(x) + \gamma_2 \overline{P}_1(x) + \gamma_3 x^2) \, dx = \gamma_2 \text{ and}$$

$$1 = \int_{-1}^{1} (\overline{P}_2(x))^2 \, dx = \int_{-1}^{1} (\gamma_1 \overline{P}_0(x) + \gamma_3 x^2)^2 \, dx = \gamma_1^2 + \frac{2}{5} \gamma_3^2 + \frac{2\gamma_1 \gamma_3}{\sqrt{2}} \frac{2}{3} \text{ therefore}$$

$$\gamma_3 = \pm \sqrt{\frac{5}{2}} \frac{3}{2} \,,\; \gamma_1 = \mp \sqrt{\frac{5}{2}} \frac{1}{2} \text{ and } \overline{P}_2(x) = \sqrt{\frac{5}{2}} \frac{3x^2 - 1}{2}$$

$$\text{continuing the } G-S \text{ procedure} \qquad \overline{P}_3(x) = \sqrt{\frac{7}{2}} \frac{5x^3 - 3x}{2} \text{ and } \overline{P}_4(x) = \sqrt{\frac{9}{2}} \frac{35x^4 - 30x^2 + 3}{8}$$

PROBLEM 3 using the generating function $\dfrac{1}{\sqrt{1 - 2tx + t^2}} = \sum_{l=0}^{\infty} t^l P_l(x)$

$$P_0(x) = \left(\frac{1}{\sqrt{1 - 2tx + t^2}} \right)_{t=0} = 1, \; P_1(x) = \left(\frac{d \frac{1}{\sqrt{1-2tx+t^2}}}{dt} \right)_{t=0} = \left(\frac{-t + x}{(1 + t^2 - 2tx)^{3/2}} \right)_{t=0} = x$$

$$P_2(x) = \frac{1}{2!} \left(\frac{d^2 \frac{1}{\sqrt{1-2tx+t^2}}}{dt^2} \right)_{t=0} = \frac{1}{2} \left(\frac{-1 + 2t^2 - 4tx + 3x^2}{(1 + t^2 - 2tx)^{5/2}} \right)_{t=0} = \frac{3x^2 - 1}{2}$$

$$P_3(x) = \frac{1}{3!} \left(\frac{d^3 \frac{1}{\sqrt{1-2tx+t^2}}}{dt^3} \right)_{t=0} = \frac{1}{6} \left(\frac{-3(2t^3 + 3x - 6t^2 x - 5x^3 + t(-3 + 9x^2))}{(1 + t^2 - 2tx)^{7/2}} \right)_{t=0} = \frac{5x^3 - 3x}{2}$$

$$P_4(x) = \frac{1}{4!} \left(\frac{d^4 \frac{1}{\sqrt{1-2tx+t^2}}}{dt^4} \right)_{t=0} =$$

$$\frac{1}{24} \left(\frac{3(3 + 8t^4 - 32t^3 x - 30x^2 + 35x^4 + 24t^2(-1 + 3x^2) + t(48x - 80x^3))}{(1 + t^2 - 2tx)^{9/2}} \right)_{t=0} = \frac{35x^4 - 30x^2 + 3}{8}$$

PROBLEM 4 using the Rodrigues formula $P_n(x) = \dfrac{1}{2^n\, n!}\dfrac{d^n}{d x^n}(x^2 - 1)^n$ obviously

$P_0(x) = 1$, $P_1(x) = x$, $P_2(x) = \dfrac{3x^2 - 1}{2}$, $P_3(x) = \dfrac{5x^3 - 3x}{2}$, $P_4(x) = \dfrac{35x^4 - 30x^2 + 3}{8}$