Catalin Constantinide

PHYS 600 – HW 10 SOLUTIONS

PROBLEM 1 (*a*) consider the function $f(z)$ defined by $f(z) = \dfrac{z^3 + 2z^2 + 4}{(z-1)^3}$ .The Laurent expansion around $z = 1$ is

$f(z) = \displaystyle\sum_{n=-\infty}^{\infty} a_n (z-1)^n$ with $a_n = \dfrac{1}{2\pi i}\displaystyle\oint_C \dfrac{f(u)}{(u-1)^{n+1}}\, du$, C being a contour enclosing the point $z = 1$

It results that $a_n = \dfrac{1}{2\pi i}\displaystyle\oint_C \dfrac{u^3 + 2u^2 + 4}{(u-1)^{n+4}}\, du = \dfrac{1}{2\pi i}\displaystyle\oint_C \dfrac{(u-1)^3 + 5(u-1)^2 + 7(u-1) + 7}{(u-1)^{n+4}}\, du =$

$$\frac{1}{2\pi i}\oint_C \left(\frac{1}{(u-1)^{n+1}} + \frac{5}{(u-1)^{n+2}} + \frac{7}{(u-1)^{n+3}} + \frac{7}{(u-1)^{n+4}}\right) du$$

but $\dfrac{1}{2\pi i}\displaystyle\oint_C \dfrac{du}{(u-1)^n} = 1$ if $n = 1$ and $\dfrac{1}{2\pi i}\displaystyle\oint_C \dfrac{du}{(u-1)^n} = 0$ if n is an integer other than 1, therefore

$a_n = 1$ if $n = 0$

$a_n = 5$ if $n = -1$

$a_n = 7$ if $n = -2$

$a_n = 7$ if $n = -3$

$a_n = 0$ if n is an integer other than $0, -1, -2, -3$

Hence $f(z) = \dfrac{z^3 + 2z^2 + 4}{(z-1)^3} = 1 + 5(z-1)^{-1} + 7(z-1)^{-2} + 7(z-1)^{-3}$

(*b*) $f(z) = \dfrac{z^3 + 2z^2 + 4}{(z-1)^3} = \dfrac{z^3 - 3z^2 + 3z - 1 + 5z^2 - 10z + 5 + 7z - 7 + 7}{(z-1)^3} = 1 + \dfrac{5}{z-1} + \dfrac{7}{(z-1)^2} + \dfrac{7}{(z-1)^3}$

PROBLEM 2 (*a*) evaluate $I_1 = \displaystyle\oint_C \dfrac{e^z\, dz}{z - \frac{i\pi}{2}}$, C being the boundary of *a* square with sides $x = \pm 2$, $y = \pm 2$

The boundary obviously encloses the pole $z = \dfrac{i\pi}{2}$, therefore $I_1 = 2\pi i\, e^{\frac{i\pi}{2}} = 2\pi i\, i = -2\pi$

(*b*) $I_2 = \displaystyle\oint_C \dfrac{e^{(z-\pi)^2}\, dz}{z - \frac{i\pi}{2}}$, C being the circle $|z| = 1$.

$\dfrac{\pi}{2} = 1.57 .. > 1$ so C does not enclose the pole $z = \dfrac{i\pi}{2} > 1$. Therefore $I_2 = 0$

(*c*) $I_3 = \displaystyle\int_0^\infty \dfrac{x^2\, dx}{(x^2 + 2)^2}$ .Changing the variable of integration $x = \sqrt{2}\tan\theta$, $dx = \dfrac{\sqrt{2}\, d\theta}{\cos^2\theta}$

$$I_3 = \int_0^{\frac{\pi}{2}} \frac{2\tan^2\theta}{(2 + 2\tan^2\theta)^2}\, \frac{\sqrt{2}\, d\theta}{\cos^2\theta} = \frac{1}{\sqrt{2}}\int_0^{\frac{\pi}{2}} \sin^2\theta\, d\theta = \frac{\pi}{4\sqrt{2}}$$

(*d*) $I_4 = \displaystyle\int_0^{2\pi} \dfrac{e^{i\theta}\, d\theta}{b + \cos\theta}$ with $b > 1$. Changing the variable of integration $z = e^{i\theta}$, $dz = i\, e^{i\theta}\, d\theta$ and $\cos\theta = \dfrac{1}{2}\left(z + \dfrac{1}{z}\right)$

$I_4 = \displaystyle\oint_C \dfrac{dz}{i}\, \dfrac{1}{b + \frac{1}{2}(z + \frac{1}{z})} = \dfrac{1}{i}\displaystyle\oint_C \dfrac{2z\, dz}{z^2 + 2bz + 1} = \dfrac{1}{i}\displaystyle\oint_C \dfrac{2z\, dz}{(z - z_+)(z - z_-)}$, C being the unit circle in the complex plane and $z_\pm$ being the roots of the equation $z^2 + 2bz + 1 = 0$, $z_\pm = -b \pm \sqrt{b^2 - 1}$

It ' *s* obvious that $-b - \sqrt{b^2 - 1} < -1$ if $b > 1$ therefore z_- is not enclosed by C

while $-1 < -b + \sqrt{b^2 - 1} < 0$ if $b > 1$ therefore z_+ is enclosed by C

Apply again the residues theorem $I_4 = \dfrac{1}{i}\, 2\pi i\, \dfrac{2z_+}{z_+ - z_-} = 2\pi\, \dfrac{2\left(-b + \sqrt{b^2 - 1}\right)}{2\sqrt{b^2 - 1}} = 2\pi\left(1 - \dfrac{b}{\sqrt{b^2 - 1}}\right)$