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PHYS 600 – HW 9 SOLUTIONS

PROBLEM 1 consider the function $f(z)$ defined by $f(z) = \frac{x y^2 (x + i y)}{x^2 + y^4}$ if $z = x + i y \neq 0$ and $f(z) = 0$ if $z = x + i y = 0$

Then $f(z) = u(x, y) + i v(x, y)$ with $u(x, y) = \frac{x^2 y^2}{x^2 + y^4}$ if $x + i y \neq 0$ and $u(x, y) = 0$ if $x + i y = 0$

$v(x, y) = \frac{x y^3}{x^2 + y^4}$ if $x + i y \neq 0$ and $v(x, y) = 0$ if $x + i y = 0$

Solving the Cauchy – Riemann equations $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ and $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$ for $x^2 + y^4 \neq 0$, one obtain that

$\frac{2 x y^6}{(x^2 + y^4)^2} = \frac{3 x^3 y^2 - x y^6}{(x^2 + y^4)^2}$ and $\frac{2 x^2 y (x^2 - y^4)}{(x^2 + y^4)^2} = \frac{y^3 (x^2 - y^4)}{(x^2 + y^4)^2}$ therefore $x = \pm y^2$ or $y = 0$

It results that the three curves $x = \pm y^2$ and $y = 0$ contain the points where $f(z)$ is analytic

$f(z)$ is not analytic at the origin because $\frac{f(z) - f(0)}{z - 0} = \frac{\frac{x y^2 (x + i y)}{x^2 + y^4} - 0}{x + i y - 0} = \frac{x y^2}{x^2 + y^4}$ and ,

taking $x = \alpha y^2$ with α being an arbitrary constant $\frac{f(z) - f(0)}{z - 0} = \frac{\alpha y^4}{\alpha^2 y^4 + y^4} = \frac{\alpha}{1 + \alpha^2}$; therefore the limit

$\lim_{z \to 0} \frac{f(z) - f(0)}{z - 0}$ does not exist $\left(\text{it would be } \frac{1}{2} \text{ if } \alpha = 1 \text{ and } -\frac{1}{2} \text{ if } \alpha = -1 \text{ for example} \right)$

PROBLEM 2 consider the function $u = \sin x \cosh y + 2 \cos x \sinh y + x^2 + 4 x y - y^2$

a simple calculation will show that $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$

the conjugate harmonic of u is a function v defined by the Cauchy – Riemann equations

$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ and $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$ therefore $\frac{\partial v}{\partial x} = -4 x + 2 y - 2 \cos x \cosh y - \sin x \sinh y$

and $\frac{\partial v}{\partial y} = 2 x + 4 y + \cos x \cosh y - 2 \sin x \sinh y$

Integrating the first differential equations $w.r.t.$ x

$v = -2 x^2 + 2 x y - 2 \sin x \cosh y + \cos x \sinh y + g(y)$, $g(y)$ being a function to be determined

Replace the last result into the second eq

$2 x + 4 y + \cos x \cosh y - 2 \sin x \sinh y = \frac{\partial v}{\partial y} = 2 x - 2 \sin x \sinh y + \cos x \cosh y + g'(y)$

hence $g'(y) = 4 y$, $g(y) = 2 y^2$ and $v = -2 x^2 + 2 x y + 2 y^2 - 2 \sin x \cosh y + \cos x \sinh y$

$f(z) = u + i v = \sin x \cosh y + 2 \cos x \sinh y + x^2 + 4 x y - y^2 + i (-2 x^2 + 2 x y + 2 y^2 - 2 \sin x \cosh y + \cos x \sinh y) =$

$(1 - 2 i)(\sin(x + i y) + x^2 - y^2 + 2 i x y) = (1 - 2 i)(\sin z + z^2)$

PROBLEM 3 let $f(z) = 2 z + 3 z^2$ and C be the unit circle around the origin

obviously $d z = d(e^{i\theta}) = i e^{i\theta} d\theta$

$$\oint_C f(z) \, dz = \int_0^{2\pi} (2 e^{i\theta} + 3 e^{2 i\theta}) i e^{i\theta} d\theta = \int_0^{2\pi} (2 i e^{2 i\theta} + 3 i e^{3 i\theta}) d\theta = (e^{2i\theta} + e^{3i\theta})_{|\theta = 2\pi} - (e^{2i\theta} + e^{3i\theta})_{|\theta = 0} = 0$$

PROBLEM 4 consider the following points in the complex plane $(z = x + i y)$

$O = 0$, $A = 2$, $B = 2 + i$

then $x = 2 y$ on the path $C_1 = OB$ and $f(z) = \bar{z} = x - i y$

$$\int_{OB} f(z)\,dz = \int_{OB} (x - iy)\; d(x + iy) = \int_0^1 (2y - iy)\; d(2y + iy) = (2 - i)(2 + i)\int_0^1 y\,dy = \frac{5}{2}$$

$y = 0$ on the path OA therefore $\displaystyle\int_{OB} f(z)\,dz = \int_{OA} (x - iy)\; d(x + iy) = \int_0^2 x\;dx = 2$

$x = 2$ on the path AB therefore $\displaystyle\int_{AB} f(z)\,dz = \int_{AB} (x - iy)\; d(x + iy) = \int_0^1 (2 - iy)\; d(2 + iy) = 2i + \frac{1}{2}$

hence $\displaystyle\int_{C_2} f(z)\,dz = \int_{OA+AB} f(z)\,dz = 2i + \frac{5}{2} \neq \frac{5}{2} = \int_{C_1} f(z)\,dz$