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PHYS 600 – HW4 SOLUTIONS

PROBLEM 1 (*a*) replace $d\xi^\alpha = \frac{\partial \xi^\alpha}{\partial x^\mu} dx^\mu$ and $d\xi^\beta = \frac{\partial \xi^\beta}{\partial x^\nu} dx^\nu$ inside $ds^2 = \eta_{\alpha\beta} d\xi^\alpha d\xi^\beta$ to obtain

$ds^2 = \eta_{\alpha\beta} \frac{\partial \xi^\alpha}{\partial x^\mu} \frac{\partial \xi^\beta}{\partial x^\nu} dx^\mu dx^\nu$ so we can identify $g_{\mu\nu}(x) = \eta_{\alpha\beta} \frac{\partial \xi^\alpha}{\partial x^\mu} \frac{\partial \xi^\beta}{\partial x^\nu}$

The inverse metric is $g^{\mu\nu}(x) = \eta^{\alpha\beta} \frac{\partial x^\mu}{\partial \xi^\alpha} \frac{\partial x^\nu}{\partial \xi^\beta}$

(*b*) we'll use the definition of the affine connection $\Gamma^\lambda_{\mu\nu} = \frac{\partial x^\lambda}{\partial \xi^\alpha} \frac{\partial^2 \xi^\alpha}{\partial x^\mu \partial x^\nu}$ and the result from (*a*); then

$$\frac{\partial g_{\mu\nu}}{\partial x^\lambda} = \frac{\partial}{\partial x^\lambda} \eta_{\alpha\beta} \frac{\partial \xi^\alpha}{\partial x^\mu} \frac{\partial \xi^\beta}{\partial x^\nu} = \eta_{\alpha\beta} \frac{\partial^2 \xi^\alpha}{\partial x^\lambda \partial x^\mu} \frac{\partial \xi^\beta}{\partial x^\nu} + \eta_{\alpha\beta} \frac{\partial \xi^\alpha}{\partial x^\mu} \frac{\partial^2 \xi^\beta}{\partial x^\lambda \partial x^\nu}$$

$$\frac{\partial g_{\lambda\nu}}{\partial x^\mu} = \frac{\partial}{\partial x^\mu} \eta_{\alpha\beta} \frac{\partial \xi^\alpha}{\partial x^\lambda} \frac{\partial \xi^\beta}{\partial x^\nu} = \eta_{\alpha\beta} \frac{\partial^2 \xi^\alpha}{\partial x^\mu \partial x^\lambda} \frac{\partial \xi^\beta}{\partial x^\nu} + \eta_{\alpha\beta} \frac{\partial \xi^\alpha}{\partial x^\lambda} \frac{\partial^2 \xi^\beta}{\partial x^\mu \partial x^\nu}$$

$$-\frac{\partial g_{\mu\lambda}}{\partial x^\nu} = -\frac{\partial}{\partial x^\nu} \eta_{\alpha\beta} \frac{\partial \xi^\alpha}{\partial x^\mu} \frac{\partial \xi^\beta}{\partial x^\lambda} = -\eta_{\alpha\beta} \frac{\partial^2 \xi^\alpha}{\partial x^\nu \partial x^\mu} \frac{\partial \xi^\beta}{\partial x^\lambda} - \eta_{\alpha\beta} \frac{\partial \xi^\alpha}{\partial x^\mu} \frac{\partial^2 \xi^\beta}{\partial x^\nu \partial x^\lambda}$$

adding the last three equalities $\frac{\partial g_{\mu\nu}}{\partial x^\lambda} + \frac{\partial g_{\lambda\nu}}{\partial x^\mu} - \frac{\partial g_{\mu\lambda}}{\partial x^\nu} = \eta_{\alpha\beta} \frac{\partial^2 \xi^\alpha}{\partial x^\lambda \partial x^\mu} \frac{\partial \xi^\beta}{\partial x^\nu} + \eta_{\alpha\beta} \frac{\partial^2 \xi^\alpha}{\partial x^\mu \partial x^\lambda} \frac{\partial \xi^\beta}{\partial x^\nu} = 2\, \eta_{\alpha\beta} \frac{\partial^2 \xi^\alpha}{\partial x^\lambda \partial x^\mu} \frac{\partial \xi^\beta}{\partial x^\nu}$

Finally $\frac{1}{2} g^{\nu\sigma} \left(\frac{\partial g_{\mu\nu}}{\partial x^\lambda} + \frac{\partial g_{\lambda\nu}}{\partial x^\mu} - \frac{\partial g_{\mu\lambda}}{\partial x^\nu} \right) =$

$$\frac{1}{2} \eta^{\rho\tau} \frac{\partial x^\nu}{\partial \xi^\rho} \frac{\partial x^\sigma}{\partial \xi^\tau} 2\, \eta_{\alpha\beta} \frac{\partial^2 \xi^\alpha}{\partial x^\lambda \partial x^\mu} \frac{\partial \xi^\beta}{\partial x^\nu} = \eta^{\rho\tau} \eta_{\alpha\beta} \delta^\beta_\rho \frac{\partial x^\sigma}{\partial \xi^\tau} \frac{\partial^2 \xi^\alpha}{\partial x^\lambda \partial x^\mu} = \delta^\tau_\alpha \frac{\partial x^\sigma}{\partial \xi^\tau} \frac{\partial^2 \xi^\alpha}{\partial x^\lambda \partial x^\mu} = \frac{\partial x^\sigma}{\partial \xi^\alpha} \frac{\partial^2 \xi^\alpha}{\partial x^\lambda \partial x^\mu} = \Gamma^\sigma_{\mu\nu}$$

PROBLEM 2 (*a*) consider in 3 – dimensions $X^{rst} = \frac{\partial x'^r}{\partial x^a} \frac{\partial x'^s}{\partial x^b} \frac{\partial x'^t}{\partial x^c} \epsilon^{abc}$, obviously X^{rst} is totally antisymmetric and

$$X^{123} = \frac{\partial x'^1}{\partial x^a} \frac{\partial x'^2}{\partial x^b} \frac{\partial x'^3}{\partial x^c} \epsilon^{abc} = \det \begin{pmatrix} \frac{\partial x'^1}{\partial x^1} & \frac{\partial x'^1}{\partial x^2} & \frac{\partial x'^1}{\partial x^3} \\ \frac{\partial x'^2}{\partial x^1} & \frac{\partial x'^2}{\partial x^2} & \frac{\partial x'^2}{\partial x^3} \\ \frac{\partial x'^3}{\partial x^1} & \frac{\partial x'^3}{\partial x^2} & \frac{\partial x'^3}{\partial x^3} \end{pmatrix} \text{ so that } X^{rst} = \epsilon^{rst} X^{123} = \epsilon^{rst} \det\left(\frac{\partial x'}{\partial x}\right)$$

(*b*) the last equation can be written $\epsilon'^{rst} = \left(\det\left(\frac{\partial x'}{\partial x}\right)\right)^{-1} \frac{\partial x'^r}{\partial x^a} \frac{\partial x'^s}{\partial x^b} \frac{\partial x'^t}{\partial x^c} \epsilon^{abc}$

consider two transformations $x \to x'$ and $x \to x''$ that corresponds to $\epsilon^{abc} \to \epsilon'^{rst} = \left(\det\left(\frac{\partial x'}{\partial x}\right)\right)^{-1} \frac{\partial x'^r}{\partial x^a} \frac{\partial x'^s}{\partial x^b} \frac{\partial x'^t}{\partial x^c} \epsilon^{abc}$,

respectively $\epsilon^{abc} \to \epsilon''^{lmn} = \left(\det\left(\frac{\partial x''}{\partial x}\right)\right)^{-1} \frac{\partial x''^l}{\partial x^a} \frac{\partial x''^m}{\partial x^b} \frac{\partial x''^n}{\partial x^c} \epsilon^{abc}$; then

$$\epsilon''^{lmn} = \left(\det\left(\frac{\partial x''}{\partial x}\right)\right)^{-1} \frac{\partial x''^l}{\partial x'^r} \frac{\partial x''^m}{\partial x'^s} \frac{\partial x''^n}{\partial x'^t} \frac{\partial x'^r}{\partial x^a} \frac{\partial x'^s}{\partial x^b} \frac{\partial x'^t}{\partial x^c} \epsilon^{abc} = \left(\det\left(\frac{\partial x''}{\partial x}\right)\right)^{-1} \frac{\partial x''^l}{\partial x'^r} \frac{\partial x''^m}{\partial x'^s} \frac{\partial x''^n}{\partial x'^t} \frac{\epsilon'^{rst}}{\left(\det\left(\frac{\partial x'}{\partial x}\right)\right)^{-1}}$$

or $\epsilon''^{lmn} = \left(\det\left(\frac{\partial x''}{\partial x'}\right)\right)^{-1} \frac{\partial x''^l}{\partial x'^r} \frac{\partial x''^m}{\partial x'^s} \frac{\partial x''^n}{\partial x'^t} \epsilon'^{rst}$

(*c*) under *a* parity transformation $\vec{x} \to \vec{x}' = -\vec{x}$ and $\vec{e}_i \to \vec{e}_i' = -\vec{e}_i$ $\quad i = \overline{1, 3}$

$$\det\left(\frac{\partial x'}{\partial x}\right) = \det \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix} = -1 \text{ so } \epsilon'^{abc} = -\left(-\frac{\partial x^a}{\partial x^p}\right)\left(-\frac{\partial x^b}{\partial x^q}\right)\left(-\frac{\partial x^c}{\partial x^r}\right) \epsilon^{pqr} = \epsilon^{abc}$$

$\vec{r} = r_i \vec{e}_i \to \vec{r}' = r_i' \vec{e}_i' = \vec{r}$ so $r_i' = -r_i$ and $\vec{p} = p_i \vec{e}_i \to \vec{p}' = p_i' \vec{e}_i' = \vec{p}$ so $p_i' = -p_i$

$\vec{L} = \epsilon^{ijk}\,\vec{e}_i\, r_j\, p_k \to \vec{L}\,' = \epsilon^{ijk}\,\vec{e}\,'_i\,(-r_j)\,(-p_k) = \epsilon^{ijk}\,\vec{e}_i\,'\, r_j\, p_k = -\vec{L}$ so $\vec{L}\,' = -\vec{L}$ $\left(\vec{L} \text{ is not } a \text{ vector}\right)$

PROBLEM 3 (*a*) using the notation $x_4 = i\,c\,t$, $A_4 = i\,\frac{\phi}{c}$ and $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$

$F_{k4} = -F_{4k} = \partial_k A_4 - \partial_4 A_k = \partial_k\, i\,\frac{\phi}{c} - \frac{\partial_t A_k}{i\,c} = \frac{1}{i\,c} E_k$ and $F_{kl} = \partial_k A_l - \partial_l A_k = \epsilon_{klm}\,\epsilon_{mnp}\,\partial_n A_p = \epsilon_{klm}\, B_m$

In matrix form $F_{\mu\nu} = \begin{pmatrix} 0 & B_3 & -B_2 & \frac{1}{i\,c} E_1 \\ -B_3 & 0 & B_1 & \frac{1}{i\,c} E_2 \\ B_2 & -B_1 & 0 & \frac{1}{i\,c} E_3 \\ -\frac{1}{i\,c} E_1 & -\frac{1}{i\,c} E_2 & -\frac{1}{i\,c} E_3 & 0 \end{pmatrix}$

(*b*) the first equation can be written $\partial_k F_{k4} = \frac{4\,\pi\,\rho}{i\,c}$ or $\partial_\nu F_{\nu 4} = \frac{4\,\pi\, j_4}{c}$ and the second is equal to

$\epsilon_{ijk}\,\partial_j\,\epsilon_{klm}\,\partial_l\, A_m - \frac{i\,c}{c}\,\partial_4\; i\,c\,F_{i4} = \frac{4\,\pi\, j_i}{c}$ or $\partial_j F_{ij} + \partial_4 F_{i4} = \partial_\nu F_{i\nu} = \frac{4\,\pi\, j_i}{c}$

(*c*) define another antisymmetric tensor $\tilde{F}_{\mu\nu} = \frac{1}{2}\,\epsilon_{\mu\nu\rho\sigma}\, F_{\rho\sigma}$ with $\epsilon_{1234} = 1$, then $\tilde{F}_{12} = F_{34}$, $\tilde{F}_{23} = -F_{41}$, $\tilde{F}_{13} = -F_{24}$, etc

In matrix form $\tilde{F}_{\mu\nu} = \begin{pmatrix} 0 & \frac{1}{i\,c} E_3 & -\frac{1}{i\,c} E_2 & B_1 \\ -\frac{1}{i\,c} E_3 & 0 & \frac{1}{i\,c} E_1 & B_2 \\ \frac{1}{i\,c} E_2 & -\frac{1}{i\,c} E_1 & 0 & B_3 \\ -B_1 & -B_2 & -B_3 & 0 \end{pmatrix}$; looking at the two matrices it's obvious that

$F_{\mu\nu}\, F_{\mu\nu} = 2\left(\vec{B}^2 - \frac{\vec{E}^2}{c^2}\right)$ and $F_{\mu\nu}\,\tilde{F}_{\mu\nu} = \frac{4}{i\,c}\,\vec{E}\cdot\vec{B}$