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PHYS 600 – HW3 SOLUTIONS

PROBLEM 1 (*a*) the transformation of *a* vector $\vec{u}(\vec{x})$ (having the covariant coordinates u_μ) is $u_\beta' = \frac{\partial x^\mu}{\partial x'^\beta} u_\mu$,

while the the transformation of *a* vector $\vec{v}(\vec{x})$ (having the contravariant coordinates v^μ) is $v'^\beta = \frac{\partial x'^\beta}{\partial x^\nu} v^\nu$

Then $u_\beta' \, v'^\beta = \frac{\partial x^\mu}{\partial x'^\beta} u_\mu \frac{\partial x'^\beta}{\partial x^\nu} v^\nu = \frac{\partial x^\mu}{\partial x^\nu} u_\mu v^\nu = u_\mu v^\mu$

(*b*) if $S_{\mu\nu}(x) = S_{\nu\mu}(x)$ then $S_{\alpha\beta}'(x') = \frac{\partial x^\mu}{\partial x'^\alpha} \frac{\partial x^\nu}{\partial x'^\beta} S_{\mu\nu}(x) = \frac{\partial x^\mu}{\partial x'^\alpha} \frac{\partial x^\nu}{\partial x'^\beta} S_{\nu\mu}(x) = S_{\beta\alpha}'(x')$

(*c*) if $A_{\mu\nu}(x) = -A_{\nu\mu}(x)$ then $A_{\alpha\beta}'(x') = \frac{\partial x^\mu}{\partial x'^\alpha} \frac{\partial x^\nu}{\partial x'^\beta} A_{\mu\nu}(x) = -\frac{\partial x^\mu}{\partial x'^\alpha} \frac{\partial x^\nu}{\partial x'^\beta} A_{\nu\mu}(x) = -A_{\beta\alpha}'(x')$

PROBLEM 2 (*a*) $T'^{\ \nu}_{\mu}(x') = \frac{\partial x^\alpha}{\partial x'^\mu} \frac{\partial x'^\nu}{\partial x^\beta} T^{\ \beta}_{\alpha}(x)$

(*b*) suppose that we apply *a* second transformation $x' \to x''$. Then

$$T''^{\ \nu}_{\mu}(x'') = \frac{\partial x'^\lambda}{\partial x''^\mu} \frac{\partial x''^\nu}{\partial x'^\rho} T'^{\ \rho}_{\lambda}(x') = \frac{\partial x'^\lambda}{\partial x''^\mu} \frac{\partial x''^\nu}{\partial x'^\rho} \frac{\partial x^\alpha}{\partial x'^\lambda} \frac{\partial x'^\rho}{\partial x^\beta} T^{\ \beta}_{\alpha}(x) = \frac{\partial x^\alpha}{\partial x''^\mu} \frac{\partial x''^\nu}{\partial x^\beta} T^{\ \beta}_{\alpha}(x)$$

(if we transform directly $x \to x''$, we'll obtain the same result)

(*c*) if *T* and *S* are tensors then their components will satisfy $T'^{\ \nu}_{\mu}(x') = \frac{\partial x^\alpha}{\partial x'^\mu} \frac{\partial x'^\nu}{\partial x^\beta} T^{\ \beta}_{\alpha}(x)$ and $S'^{\ \nu}_{\mu}(x') = \frac{\partial x^\alpha}{\partial x'^\mu} \frac{\partial x'^\nu}{\partial x^\beta} S^{\ \beta}_{\alpha}(x)$

Therefore $a\,T'^{\ \nu}_{\mu}(x') + b\,S'^{\ \nu}_{\mu}(x') = \frac{\partial x^\alpha}{\partial x'^\mu} \frac{\partial x'^\nu}{\partial x^\beta} (a\,T^{\ \beta}_{\alpha}(x) + b\,S^{\ \beta}_{\alpha}(x))$

PROBLEM 3 (*a*) $\beta \equiv \frac{v}{c}$, $\gamma \equiv \frac{1}{\sqrt{1-\beta^2}}$, $\xi \equiv c\,t$, obviously $\gamma^2 - \beta^2\gamma^2 = 1$

The Lorentz transformations are

$x_1' = \gamma(x_1 - \beta\xi_1)$, $y_1' = y_1$, $z_1' = z_1$, $\xi_1' = \gamma(\xi_1 - \beta x_1)$ and $x_2' = \gamma(x_2 - \beta\xi_2)$, $y_2' = y_2$, $z_2' = z_2$, $\xi_2' = \gamma(\xi_2 - \beta x_2)$

(*b*) $(x_2' - x_1')^2 + (y_2' - y_1')^2 + (z_2' - z_1')^2 - (\xi_2' - \xi_1')^2 =$

$(\gamma(x_2 - x_1) - \beta\gamma(\xi_2 - \xi_1))^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2 - (\gamma(\xi_2 - \xi_1) - \beta\gamma(x_2 - x_1))^2$

$= (\gamma^2 - \beta^2\gamma^2)(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2 - (\gamma^2 - \beta^2\gamma^2)(\xi_2 - \xi_1)^2 = (x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2 - (\xi_2 - \xi_1)^2$

(*c*) if $\beta \equiv \tanh(u)$ then $\gamma = \frac{1}{\sqrt{1 - \tanh^2(u)}} = \cosh(u)$ and the Lorentz transformations become

$x_1' = \cosh(u)\,x_1 - \sinh(u)\,\xi_1$, $y_1' = y_1$, $z_1' = z_1$, $\xi_1' = \cosh(u)\,\xi_1 - \sinh(u)\,x_1$

(*d*) using the notation $x_4 = i\,c\,t = i\,\xi$ and $x^\mu = (x, y, z, x_4)^t$

$x' = \cosh(u)\,x + i\,\sinh(u)\,x_4$, $y' = y$, $z' = z$, $x_4' = \cosh(u)\,x_4 - i\,\sinh(u)\,x_1$

In matrix notation $x'^\mu = \Lambda^\mu_\nu x^\nu$ or

$$\begin{pmatrix} x' \\ y' \\ z' \\ x_4' \end{pmatrix} = \begin{pmatrix} \cosh(u) & 0 & 0 & i\,\sinh(u) \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -i\,\sinh(u) & 0 & 0 & \cosh(u) \end{pmatrix} \begin{pmatrix} x \\ y \\ z \\ x_4 \end{pmatrix} = \begin{pmatrix} \cos(i\,u) & 0 & 0 & \sin(i\,u) \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -\sin(i\,u) & 0 & 0 & \cos(i\,u) \end{pmatrix} \begin{pmatrix} x \\ y \\ z \\ x_4 \end{pmatrix}$$