Physics Graduate School Qualifying Examination

Spring 2014  Part I

Instructions: Work all problems. This is a closed book examination. Calculators may not be used. **Start each problem on a new pack of correspondingly numbered paper and use only one side of each sheet.** Place your 3-digit identification number in the upper right hand corner of each and every answer sheet. All sheets, which you receive, should be handed in, even if blank. Your 3-digit ID number is located on your envelope. All problems carry the same weight.

- Correct answers without adequate explanation/reasoning will not get full credit.
- Explain all variables you use in your derivations.
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- Use correct vector notation when appropriate.
- Your work must be legible and clear.

1) Consider a rope of length, L, and mass, M. Half of the rope is placed on a table of height, h, and the other half is left hanging off the table. Assume there is no friction on the table, h > L, and the rope does not rotate.

a) What is the equation of motion for the center of mass of the rope after it is released?

b) Find an expression for the time it takes for the rope to be fully off the table.

[Hints: a general solution to \( \frac{d^2 f}{dx^2} = a + bf \) is

\[
f = -\frac{a}{b} + c_1 e^{x\sqrt{b}} + c_2 e^{-x\sqrt{b}},
\]

and you may find it useful to use: \( \cosh(x) = \frac{1}{2}(e^x + e^{-x}) \)
2) A roller coaster designer wants to make a roller coaster follow a path such that it starts on a long ramp followed by a loop. The end of the ramp starts at a height, \( h \), above the ground, and is at an angle \( \theta \) relative to the horizon. The loop has a radius of \( R \). The coefficient of kinetic friction on the ramp is \( \mu \), and the coefficient of friction after the ramp is 0. Model the roller coaster as a sliding object of mass \( M \). Take the gravitational acceleration to be \( g \).

A) What is the velocity of the roller coaster right before it reaches the loop?

B) What is the condition on the parameters of the ramp \((h, \theta)\) so that the roller coaster will make it around the loop completely?
3) Suppose we bring in charges from infinity, and assemble them into a sphere of radius $R$, with a uniform charge density $\rho$. Compute the total work required for this process.
4) A solenoid with \( n \) turns per unit length has an inner radius \( r \) and has one end immersed vertically into a bath of liquid oxygen, a paramagnetic substance with magnetic susceptibility \( \chi_m \) and mass density \( \rho \). How much will the height of the liquid oxygen in the solenoid change when a current \( I \) flows through the coils compared with the height when no current flows? Will the liquid oxygen be sucked up or pushed down?

Take the gravitational acceleration to be \( g \).
5) Consider a quantum mechanical system consisting of particle of mass $m$ moving in a one-dimensional periodic potential given by (see also figure)

$$V(x) = \frac{\hbar^2}{4m x_0^2} \left[ \sin \frac{x}{x_0} + \frac{1}{4} \cos \frac{2x}{x_0} \right]$$

a) Find the value of $a$ such that the wave function

$$\psi(x) = e^{-a \sin \frac{x}{x_0}}$$

is an Energy eigenfunction of such potential.

b) Compute the Energy eigenvalue.

![Figure 1: Potential $V(x)$](image)
6) Consider a spin $\frac{1}{2}$ particle in a constant magnetic field along the direction $z$. A time dependent magnetic field is added in the directions $x,y$ such that the Hamiltonian of the two state spin system is given by

$$H = \begin{pmatrix} -\frac{\hbar \omega}{2} & 0 \\ 0 & +\frac{\hbar \omega}{2} \end{pmatrix} + \gamma \begin{pmatrix} 0 & e^{i\omega t} \\ e^{-i\omega t} & 0 \end{pmatrix}$$

in the basis of spin up and down in the $z$ direction. Notice the resonant condition that $\omega$ is the same for the energy split in the first term and the oscillation frequency in the second term.

a) Write the (time dependent) spin state of the system as

$$\psi = \begin{pmatrix} e^{+i\frac{\omega t}{2}} \alpha(t) \\ e^{-i\frac{\omega t}{2}} \beta(t) \end{pmatrix}$$

and find the differential equations for $\alpha(t)$ and $\beta(t)$.

b) Assume that, at time $t=0$, the spin is pointing up in the $z$ direction and find the state of the system at all times.

c) If you measure the component $z$ of the spin at an arbitrary time $t$, what are the respective probabilities of finding the spin pointing up or down in the $z$ direction?
7) Consider a closed system of $N=5$ distinguishable, independent, classical spins, each with magnetic moments $\mu_0$ in a magnetic field of magnitude $H$. The energy for spin $i$ is $U_i = -2s_i \mu_0 H$, where $s_i$ is either +1 or -1 depending on whether the spin is parallel or antiparallel with $H$. If the total energy of the system is $E = -6\mu_0 H$, what is the entropy of the system?
8) A non-relativistic electron (mass m, charge -e) is emitted with initial velocity $v_0$ parallel to an infinite grounded conducting plane. The initial distance from the plane is $y_0$.

Assume that the induced charge on the plane adjusts instantaneously to the electron's position. Neglect gravity and radiation by the electron.

a) Construct a suitable energy function (first integral) for this problem – a conserved function $E(y,dy/dt)$, where $y$ is the distance from the plane.

b) Find the time at which the electron will hit the plane. (For this part, you may find it useful to change the variable from $y$ to $\eta$, where $y = y_0 \cos^2(\eta)$.)
Useful Equations

\[ \nabla \cdot \vec{E} = \frac{\rho}{\varepsilon_0} \quad \nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} \quad i\hbar \frac{\partial \Psi}{\partial t} = H\Psi \]

\[ \nabla \cdot \vec{B} = 0 \quad \nabla \times \vec{B} = \mu_0 \vec{J} + \mu_0 \varepsilon_0 \frac{\partial \vec{E}}{\partial t} \]

\[ \vec{S} = \frac{1}{\mu_0} \vec{E} \times \vec{B} \quad \vec{B} = \nabla \times \vec{A} \quad \vec{E} = -\nabla V - \frac{\partial \vec{A}}{\partial t} \]

\[ F = -kT \ln Z \quad Z = \sum_i e^{\frac{E_i}{kT}} \quad S = -\left( \frac{\partial F}{\partial T} \right)_{V,S} \]

Lorentz Transformation Equations:

\[ x' = \gamma(x - vt) \quad t' = \gamma(t - \frac{v}{c^2} x) \quad y' = y \quad z' = z \quad \gamma = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}} \]

\[ E'_x = E_x \quad B'_x = B_x \]

\[ E'_y = \gamma(E_y - vB_z) \quad B'_y = \gamma(B_y + \frac{v}{c^2} E_z) \]

\[ E'_z = \gamma(E_z + vB_y) \quad B'_z = \gamma(B_z - \frac{v}{c^2} E_y) \]
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- Use correct vector notation when appropriate.
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1) A particle of mass $m$ starts at rest on top of a smooth fixed hemisphere of radius $R$. At which angle does the particle leave the hemisphere? Take the gravitational acceleration to be $g$. (Note that the particle slides; it does not roll.)
2) Consider the potential shown below; redraw this potential on your answer sheet and mark points of special interest accordingly.

a) Graphically construct the phase diagram for a classical particle moving in this potential, using a few representative energies (no calculations needed). Note: Phase diagram = a set of trajectories in the $x - p$ plane.

b) Under which conditions is the movement bound, under which conditions is it unbound?
3) A uniformly charged, infinitely long, insulating, cylindrical rod of radius $R$ is spinning at angular frequency $\omega$ along the $z$ axis, which is its axial symmetry axis. The charge density of the rod is $\rho$, and its relative permittivity and permeability are very close to unity, so set $\varepsilon = \varepsilon_0$ and $\mu = \mu_0$. Calculate the electric and magnetic fields everywhere (both inside and outside the rod).

The following integrals may be useful:

\[
\int_{-\infty}^{\infty} dx \frac{1}{(x^2 + a^2)^{3/2}} = \frac{2}{a^2} ,
\]

\[
\int_{0}^{2\pi} dx \frac{a - b \cos(x)}{a^2 + b^2 - 2ab \cos(x)} = \begin{cases} 
2\pi/a & \text{if } a > b > 0 \\
0 & \text{if } b > a > 0
\end{cases}
\]
4) We shine a linearly polarized monochromatic electromagnetic plane wave of angular frequency $\omega$ at a flat (planar) surface of an infinitely thick, electrically neutral material at normal incidence. We know the permittivity $\varepsilon$ and permeability $\mu$ of the material, and we also know that it is a conductor, with a very small conductivity at this frequency. Our goal is to determine the conductivity experimentally.

**Suppose we measure the phase difference $\delta \phi$ between the $E$ and $B$ fields inside the material. Express the conductivity in terms of $\delta \phi$.**

*[Hint: from Maxwell's equations, derive a relation between angular frequency and wave vector inside the medium (so-called dispersion relation).]*
5) A particle of mass $M$ is confined to the spatial region $x > 0$ and has the wave function $\psi(x) = C \cdot e^{-x/a}$ where $C$ and $a$ are real, positive constants.

(a) What are the units of $C$ and $a$?

(b) Find $C$ from the information given and whatever other fundamental constants are needed.

(c) Compute $\langle x \rangle$ and $\langle x^2 \rangle$.

(d) Compute $\langle p \rangle$ and $\langle p^2 \rangle$.

(e) Compute $\Delta x \Delta p$ and compare to the Heisenberg limit.

Possibly useful integral:

$$\int_0^\infty dx \ x^n e^{-x} = n!$$
6) A neutron of mass $M$ is trapped near the surface of the earth due to the force of gravity. (Think of the neutron as elastically bouncing off the surface.) Let $z$ denote the vertical height of the neutron, with $z=0$ being the earth’s surface. Let $g$ be the gravitational acceleration.

(a) Write down the time-independent Schrödinger equation which describes the vertical motion of the neutron, i.e. the equation for the wave function $\psi(z)$.

(b) Using the most reasonable approximation, find an expression for the energy eigenvalues when the number of nodes in the wave function is large.
7) Consider two containers, one containing \( n_A \) molecules of an ideal gas species A and the other containing \( n_B \) molecules of a different ideal gas species B. Each container is thermally isolated, and has volume \( V \). Each gas has the same energy per particle \( u \). The two containers are then connected by a small tube.

When the system reaches equilibrium, will the two gases be mixed? Prove your answer using a thermodynamic argument.
8) Isotherms of a "real" gas (that is, a gas with interactions) have, in the pressure vs. volume plane, horizontal (constant pressure) segments corresponding to liquid-gas equilibrium. Suppose two such segments, with slightly different temperatures $T$ and $T - dT$, are used in a Carnot cycle.

(a) Suppose one mole of water evaporates completely at constant temperature $T$ and constant pressure $P$; in the process it expands by an amount $V_{\text{gas}} - V_{\text{liquid}}$. Then, it is cooled adiabatically to temperature $T - dT$ and pressure $P - dP$, isothermally condensed at these new values and, finally, returned adiabatically back to the original state. Approximating the adiabats by vertical segments in the $P$ vs. $V$ plane, find the work $\delta W$ extracted from the cycle.

(b) Suppose the cycle has Carnot efficiency. Relate the work $\delta W$ to the latent heat $L$ of a mole of water. Assume $L$ is independent of temperature.

(c) From your results in (a) and (b) compute the derivative $dP/dT$ in terms of other parameters given.
Useful Equations

\[ \bar{\nabla} \cdot \bar{E} = \frac{\rho}{\varepsilon_0}, \quad \bar{\nabla} \times \bar{E} = -\frac{\partial \bar{B}}{\partial t}, \quad i\hbar \frac{\partial \Psi}{\partial t} = H\Psi \]

\[ \bar{\nabla} \cdot \bar{B} = 0, \quad \bar{\nabla} \times \bar{B} = \mu_0 \bar{J} + \mu_0 \varepsilon_0 \frac{\partial \bar{E}}{\partial t} \]

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