Instructions: Work all problems. This is a closed book examination. Calculators may not be used. **Start each problem on a new pack of correspondingly numbered paper and use only one side of each sheet.** Place your 3-digit identification number in the upper right hand corner of each and every answer sheet. All sheets, which you receive, should be handed in, even if blank. Your 3-digit ID number is located on your envelope. All problems carry the same weight.

- Correct answers without adequate explanation/reasoning will not get full credit.
- Explain all variables you use in your derivations.
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- Use correct vector notation when appropriate.
- Your work must be legible and clear.

1) On take-off, an airplane travels down the runway with constant acceleration A in the +x direction. Find an expression for the pressure $p(x)$ inside the aircraft. Assume that air is an ideal gas at constant temperature $T$, satisfying $p = \rho R^* T$, where $\rho$ is the mass density and $R^*$ is a constant.
2) Consider an arrangement of three equal masses, attached to four springs with equal spring constants $K$, all attached to a movable cart as shown.

Calculate the displacement of each mass from its at-rest equilibrium position, if the cart accelerates with constant acceleration $A$ to the right. (The masses move frictionlessly on the floor of the cart.)
(3) As shown in the figure below, a metal sphere of radius \( a \) carries charge \( Q \). It is surrounded, out to radius \( b \), by linear dielectric material of permittivity \( \varepsilon \).

Recall that \( \nabla \cdot D = \rho_f \) where \( \rho_f \) is the free charge density, and \( \varepsilon = \varepsilon_0(1 + \chi_e) \), where \( \chi_e \) is the electric susceptibility.

a) Find the potential at the center of the sphere (relative to infinity).

b) Calculate the polarization in the dielectric medium.

c) Calculate the bound surface charge on the inner and outer surfaces of the dielectric.

d) What is the energy stored in this configuration?
(4) As shown below, a metal bar of mass $m$ slides frictionlessly on two parallel conducting rails a distance $l$ apart. A resistor $R$ is connected across the rails and a uniform magnetic field $B$, pointing into the page, fills the entire region.

![Diagram of a metal bar sliding on rails with a resistor connected and a magnetic field]

$m$

a) If the bar moves to the right at speed $v$, what is the current in the resistor? In what direction does it flow?
b) What is the magnetic force on the bar? In what direction does it point?
c) If the bar starts out with speed $v_0$ at time $t=0$, and is allowed to slide, what is its speed at a later time $t$?
d) The initial kinetic energy of the bar was, of course, $1/2mv_0^2$. Check that the energy delivered to the resistor is exactly $1/2mv_0^2$. 
(5) Consider the Harmonic Oscillator Hamiltonian
\[ H = \hbar \omega \hat{a}^+ \hat{a}, \]
where \( \hat{a}^+ \), \( \hat{a} \) are raising and lowering operators respectively. At \( t=0 \), the system is in a coherent state, \( |\psi> = e^{i\theta \hat{a}^+} |0> \), where \( |0> \) is the ground state of the Harmonic Oscillator and \( \phi_0 \) is a constant. Recall that \([a, a^+] = 1\)

Show that the state at time \( t > 0 \) can be written as, \( |\psi(t) >= e^{i\theta t} \hat{a}^+ |0> \) and determine \( \phi(t) \)
(6) Consider a particle with mass \( m \) is in an one-dimensional infinite quantum well with a \( \delta \)-function potential at the center of quantum well, i.e. the potential is given by

\[
V(x) = \infty \quad \text{for} \quad |x| \geq a
\]

\[
V(x) = V_0 \delta(x) \quad \text{for} \quad |x| < a
\]

Find all the energy eigenstates which have energies that are independent of \( V_0 \)
7) Suppose we have a gas of noninteracting atoms in an external magnetic field $B$, and at temperature $T$. Each atom can be in one of two states, whose energies differ by an amount $\Delta E = 2\mu B$. Here $\mu$ is a positive constant, and $B$ is also positive. Each atom’s magnetization is taken to be $+1$ if it is the lower energy state, and $-1$ if it is in the higher energy state.

a) Find the average magnetization of the entire sample as a function of $B$.

b) Discuss how your solution behaves when $B \to 0$ and when $B \to \infty$. 
(8) Consider a simple model for the surface temperature $T$ of an airless planet. Suppose the sun emits electromagnetic energy at a rate $L$, in Joules per second. The spherical planet has radius $R$, and is at a distance $z$ from the sun (with $z \gg R$) in a circular orbit. The planet is a perfect black-body, absorbing all incident radiation. The planet’s surface has the same temperature at all locations (it spins rapidly enough for this to be true). The planet emits radiation outward according to the law:

$$P = \sigma T^4 \text{ (Joules per second per square meter)}$$

Compute the equilibrium temperature $T_0$ of the planet.
Useful Equations

\[ \nabla \cdot \vec{E} = \frac{\rho}{\varepsilon_0} \quad \nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} \quad \frac{i\hbar}{\partial t} \frac{\partial \Psi}{\partial t} = H\Psi \]

\[ \nabla \cdot \vec{B} = 0 \quad \nabla \times \vec{B} = \mu_0 \vec{J} + \mu_0 \varepsilon_0 \frac{\partial \vec{E}}{\partial t} \]

\[ F = -kT \ln Z \quad Z = \sum_i^e \frac{E_i}{\sigma} \]

Lorentz Transformations Equations:

\[ x' = \gamma(x - vt) \quad t' = \gamma(t - \frac{v}{c^2} x) \quad y' = y \quad z' = z \quad \gamma = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}} \]

\[ E'_x = E_x \quad B'_x = B_x \]

\[ E'_y = \gamma(E_y - vB_z) \quad B'_y = \gamma(B_y + \frac{v}{c^2} E_z) \]

\[ E'_z = \gamma(E_z + vB_y) \quad B'_z = \gamma(B_z - \frac{v}{c^2} E_y) \]
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(1) Consider a rigid body rotating around its center of mass. The moments of inertia with respect to its principal axes are $I_1$, $I_2$ and $I_3$, and are all different. If the rigid body is rotating around axis 3 with angular velocity $\omega_3=\omega$,

(a) use the Euler equations

\[
\begin{align*}
I_1 \dot{\omega}_1 &= \omega_2 \omega_3 (I_2 - I_3) \\
I_2 \dot{\omega}_2 &= \omega_1 \omega_3 (I_3 - I_1) \\
I_3 \dot{\omega}_3 &= \omega_1 \omega_2 (I_1 - I_2)
\end{align*}
\]

to show that the motion is stable (under small perturbations) when $I_3$ is the largest or smallest moment of inertia and unstable otherwise.

(b) For the stable cases compute the precession frequency for small perturbations of the motion.
(2) Consider a rocket of total mass $M$, of which half is fuel: $M_f = M/2$. The fuel is consumed at a rate of $\dot{\rho}$ (in Kg/s) and expelled at a relative velocity $V_e$.

(a) Use momentum conservation in a diagram like in the figure to find the equation for the velocity of the rocket as a function of time.

(b) If starting at rest in empty space (no gravity), what is the maximum velocity $V_M$ that the rocket can attain, that is, when it runs out of fuel? Write your answer in terms of $V_e$.

(c) Suppose that, starting at rest, the rocket should reach a velocity $V < V_M$. Give a formula for how much of its fuel would be consumed for the rocket to reach such a velocity. (The rocket still starts with $M_f = M/2$.)
(3) The figure below shows an array of very thin parallel wires each with a charge per unit length, \( \lambda \), lying in the \( xy \)-plane. Consider the wires to be infinitely long and uniformly spaced by a distance \( a \) between them. Thus we take the charge density to be:

\[
\rho(x,z) = \lambda \delta(z) \sum_n \delta(x - na)
\]

(a) Find the electric potential \( \phi(x,z) \) from this arrangement of line charges.

(b) Show that the electric field far from the array (large \( z \)) is equivalent to that of a uniformly charged sheet of surface charge density \( \lambda/a \).

[Hint: Assume the electric potential takes the form:

\[
\phi(x,z) = c_0 F_0(z) + \sum_{n=0} \frac{c_n F_n(z) \cos\left(\frac{2\pi nx}{a}\right)}{a}.
\]

At the surface \( z=0 \), the surface charge density \( \lambda \sum_n \delta(x - na) \) is related to a discontinuity in \( B_z \). Use this fact to integrate over an interval in \( x \) from \(-a/2\) to \( a/2\), i.e., so that only one line charge contributes.] Useful integral:

\[
\int_0^\pi \cos^2 mx \, dx = \frac{\pi}{2}.
\]
(4) An electron is initially at rest. At time \( t_1 = 0 \) it is accelerated upward in the y-direction with an acceleration of \( 10^{18} \text{ m/s}^2 \) for a very short time (less than \( 10^{-12} \) seconds). Observations are made at location A, 15 meters from the electron along the x-axis. Recall that \( \vec{E}_{\text{rad}} = -\frac{1}{4\pi\varepsilon_0} \frac{q\hat{n} \times (\hat{n} \times \vec{a})}{c^2 r} \), where \( \hat{n} \) is the unit vector from the source to the observation point.

\[
a = 10^{18} \text{ m/s}^2
\]

\[\uparrow\]

Electron

15 m

A

a) At a time \( t_2 = 10^{-9} \text{ seconds} \), what is the magnitude and direction of the electric field at location A due to the electron?

b) At what time \( t_3 \) does the electric field at location A change?

c) What is the direction and magnitude of the radiative electric field at time \( t_3 \)?

d) Just after time \( t_3 \), what is the direction of the magnetic force on a positive charge that was initially at rest at location A?

Note that \( c = 3 \times 10^8 \text{ m/sec} \)
(5) An electron moves freely in a one dimensional infinite potential well with walls at $x = 0$ and $x = a$.

a) Find the ground state energy eigenfunction $\phi_1(x)$ of the electron.

b) The electron is initially in the ground state of the well and then the well is suddenly expanded by a factor of 4 so that the right hand wall is now at $x = 4a$. Find an expression for the probability that the electron will be found in the ground state $\psi_1(x)$ of the new, expanded well, in terms of the appropriate wave functions.
(6) A quantum particle of mass $m$ is constrained to move freely on the surface of a sphere of radius $R$.

a) What dynamical properties of the particle, such as perhaps energy, are conserved? Explain.

b) Give an operator equation for the Hamiltonian in terms of operators for the conserved quantities. Define your symbols.

c) Give a complete set of energy eigenfunctions of the particle. They need not be normalized.

d) Give an expression for the energy eigenvalues in terms of quantum numbers for the conserved quantities.
(7) Suppose you want to devise a cyclic process which removes an amount of heat $Q$ from a thermal reservoir at temperature $T_1$, and releases this heat to a reservoir at temperature $T_2$ (with $T_2 > T_1$). What is the minimum work $W$ required to accomplish this task?
(8) A $K_S$ meson decays into two $\pi$ mesons, i.e. $K_S \rightarrow \pi + \pi$. Take the rest mass of the $K_S$ to be $M_{K_S}=500$ MeV/c$^2$, and of the $\pi$ to be $m_\pi=0.3M_{K_S}$. The decay is isotropic in the $K_S$ rest frame. In a laboratory a beam of $K_S$ mesons moves with velocity $v=0.6c$ (c is the speed of light) in the positive $x$ direction.

a) Find the maximum energy an emitted $\pi$ meson can have, as measured in the lab system.

b) As measured in the $K_S$'s rest frame, find the rate at which the distance between the two decay $\pi$ mesons increases with time.

c) Assume that a particular $K_S$ in the beam lives for a time $\tau$ as measured in the lab system. How long will that $K_S$ live as measured by an observer traveling in the positive $z$ direction (with respect to the lab) with velocity $v_{obs}=0.5c$?

You do not have to evaluate numerical answers. You may express your results in terms of $M_{K_S}$, $m_\pi$, $v$, $v_{obs}$, $\tau$. 
Useful Equations

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