Physics Graduate School Qualifying Examination

Fall 2012  Part I

Instructions: Work all problems. This is a closed book examination. Calculators may not be used. **Start each problem on a new pack of correspondingly numbered paper and use only one side of each sheet.** Place your 3-digit identification number in the upper right hand corner of each and every answer sheet. **All sheets, which you receive, should be handed in, even if blank.** Your 3-digit ID number is located on your envelope. All problems carry the same weight.

- Correct answers without adequate explanation/reasoning will not get full credit.
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- Use correct vector notation when appropriate.
- Your work must be legible and clear.

(1) A student whose mass is $m$ and whose height is $h$ jumps vertically up from the “squat” position. In the squat position, the student’s center of mass is a height $h/4$ from the floor. At the top point of the jump, the student’s center of mass is at a height $3h/4$ from the ground. The student’s center of mass is a height $h/2$ when standing up straight on the floor. Find the force $F$ acting on the floor prior to the moment when the student loses contact with the floor. (Treat the force as a constant during the time interval over which it acts.)
(2) A uniform rod (length $3L/2$, mass $M$) is suspended from a ceiling by a massless string of length $L$. Find both the normal modes and their frequencies, for small oscillations in the 2-dimensional plane.
3) Find the charge density \( \rho \) and current density \( \vec{J} \) at the time \( t = a/c \) at all locations in the \( z = a \) plane, if the electric and magnetic fields are given by:

\[
E = E_0 \mathbf{z} \exp[- \frac{z^2}{a^2} (ct)^2 / a^4] \\
B = B_0 \mathbf{x} \exp(- \frac{z^2}{a^2})
\]

( \( \mathbf{x} \) and \( \mathbf{z} \) are unit vectors along the \( x \) and \( z \) directions, respectively; \( c \) is the speed of light, and \( a \) is a characteristic length scale). Make sure to declare which E&M convention you follow for units (e.g., write down Coulomb’s law).
(4) Consider the standard problem of reflection-refraction at a planar interface for a monochromatic plane wave.

Let the interface be the $x$-$y$ plane (given by $z=0$), and consider normal incidence with the incoming wave traveling in the $+z$ direction; the region $z < 0$ is vacuum. In the $z < 0$ region we have an incident wave and a reflected wave:

$$\vec{E}_{\text{incident}} = \hat{x} E_i \cos(kz - \omega t) \quad \vec{E}_{\text{reflected}} = \hat{x} E_r \cos(-kz - \omega t)$$

Compute the Poynting vector in the $z < 0$ region, and show that the cross term vanishes, so that

$$\vec{S} = \vec{S}_{\text{incident}} + \vec{S}_{\text{reflected}}$$

where $\vec{S}_{\text{incident}}$ is the Poynting vector due to the incident wave alone,

and $\vec{S}_{\text{reflected}}$ is the Poynting vector due to the reflected wave alone.
(5) Suppose you have a particle of mass $m$ trapped in a two-dimensional infinite square well where the potential is $V(x,y) = 0$ if $0 < x < a$ and $0 < y < a$; and $V(x,y) = \infty$ otherwise.

a) Solve the time-independent two-dimensional Schrödinger equation to find the complete set of normalized energy eigenstates.

b) Compute the corresponding energies of these eigenstates.

c) If you had two particles that were distinguishable (but with the same mass $m$) occupying this potential, write down the complete set of two-particle energy eigenstates. Assume the particles do not interact with each other.

d) If you had two non-interacting particles that were indistinguishable and were fermions (say, two electrons in a spin triplet state), write down the complete set of two-particle energy eigenstates.
(6) Consider the Hamiltonian

\[ H = \frac{p^2}{2m} + \lambda x^4 \]

that describes a particle of mass \( m \) in a one dimensional potential \( V = \lambda \cdot x^4 \).

The purpose of this problem is to find an approximate value for the ground state energy using the following steps:

a) Use dimensional analysis to show that the ground state energy \( E_0 \) of \( H \) can be written as

\[ E_0 = \lambda^a \left( \frac{\hbar^2}{m} \right)^b \eta_0 \]

where \( \eta_0 \) is a dimensionless number. Find the exponents \( a \) and \( b \).

b) Given a state with wave function

\[ \psi(x) = \left( \frac{\alpha}{\pi} \right)^{\frac{1}{4}} e^{-\frac{1}{2} \alpha x^2} \]

compute the expectation value of the energy \( E(\alpha) \) in such state.

c) Minimize the energy with respect to the parameter \( \alpha \). Compare the result with part a) and identify the approximate value of the number \( \eta_0 \).

d) Should the result in c) be larger or smaller than the actual ground state energy? Justify.

**HINTS:** The wave-function \( \psi(x) \) appearing in part (b) satisfies:

\[
\begin{align*}
\int_{-\infty}^{\infty} |\psi(x)|^2 \, dx &= 1 \\
\int_{-\infty}^{\infty} |\partial_x \psi(x)|^2 \, dx &= \frac{\alpha}{2} \\
\int_{-\infty}^{\infty} x^4 |\psi(x)|^2 \, dx &= \frac{3}{4\alpha^2}
\end{align*}
\]
7) Suppose you have a box in thermal equilibrium with a large thermal reservoir at constant temperature. The box contains a single substance in two-phase equilibrium, so that both liquid and gas phases are present. The volume of the box is reversibly (slowly) reduced, to the point where the box contains only liquid.

a.) Sketch the Pressure vs. Volume diagram for this isothermal process.

b) Do you expect that heat is transferred in from the reservoir to the box, or, out from the box to the reservoir during the compression?

c) What is the total change in entropy of the system and surroundings for this process?
Useful Equations

\[ \nabla \cdot \vec{E} = \frac{\rho}{\varepsilon_0} \quad \nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} \quad i\hbar \frac{\partial \Psi}{\partial t} = H\Psi \]

\[ \nabla \cdot \vec{B} = 0 \quad \nabla \times \vec{B} = \mu_0 \vec{J} + \mu_0 \varepsilon_0 \frac{\partial \vec{E}}{\partial t} \quad F = -kT \ln Z \quad Z = \sum_i e^{\frac{E_i}{kT}} \]

\[ S = \frac{1}{\mu_0} \vec{E} \times \vec{B} \quad S = -\left( \frac{\partial F}{\partial T} \right)_{V,B} \]

Lorentz Transformation Equations:

\[ x' = \gamma (x - vt) \quad t' = \gamma (t - \frac{v}{c^2} x) \quad y' = y \quad z' = z \quad \gamma = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}} \]

\[ E'_x = E_x \quad B'_x = B_x \]

\[ E'_y = \gamma (E_y - vB_z) \quad B'_y = \gamma (B_y + \frac{v}{c^2} E_z) \]

\[ E'_z = \gamma (E_z + vB_y) \quad B'_z = \gamma (B_z - \frac{v}{c^2} E_y) \]
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Fall 2012 Part II

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(1) Block A is placed on block B as shown in the figure below. Both blocks slide, without moving with respect to each other, along a frictionless horizontal surface at speed \( v_i \). Block B hits a resting block C “head-on.” After the collision, blocks B and C move together, and block A slides on top of board C and stops its motion relative to C in the position shown in the diagram. All three blocks have the same mass, size, and shape. It is known that there is no friction between blocks A and B; the coefficient of kinetic friction between blocks A and C is \( \mu_k \). Let the mass of each block be \( m \), and the length, \( L \). [Note: the velocity of the assembly after the collision is \( v_f < v_i \).]

What is the length \( L \) of each block in terms of \( m, v_i, \) and \( \mu_k \) ?
(2) Twin brothers, Homer and Ulysses, have identical heart beats: frequency 1 per second, each beat emits a pulse. Homer stays at rest at home (inertial reference frame). Ulysses travels at constant relativistic speed $\beta = v/c$ to a certain distance and returns. Neglect effects of acceleration from rest to speed $\beta$ and the sudden change of velocity when Ulysses returns. Ulysses spends time $t$ (according to his own watch) on the way out and another $t$ to return home.

a) How many pulses total does Ulysses emit?

b) On the way out how many pulses from Homer does Ulysses receive?

c) On the way back how many pulses from Homer does Ulysses receive?

d) What is the ratio of the total number of pulses Ulysses receives to that he emits?

Homer receives red-shifted pulses until time $t_2$ (according to his own watch) after which he receives blue-shifted pulses up to $t_3$, the time at which Ulysses reaches home.

e) Express $t_2$ and $t_3$ in terms of $t$ and $\beta$.

f) How many pulses does Homer receive within $t_2$?

f) How many pulses does Homer receive from $t_2$ to $t_3$?

h) What is the ratio of the number of pulses Homer receives to that he emits?
(3) Consider two uniform cylindrical conductors, each of length \( L/2 \) and cross-sectional area, \( A \), but with different conductivities, \( \sigma_1 \) and \( \sigma_2 \). Suppose they are joined end-to-end and connected to a voltage source, as shown:

\[ \text{Diagram showing two cylinders connected by a voltage source.} \]

(a) Calculate the voltage drop, \( \Delta V \), across the conductor with conductivity \( \sigma_1 \).

(b) Calculate the surface charge density that accumulates at the boundary between the two conductors.
(4) An electromagnet consists of a thin iron core with magnetic permeability, \( \mu \), which is bent into a circle of radius \( R \) with a small gap of width \( w \). If there are \( N \) turns of wire carrying a current \( I \) wrapped around the core, calculate the magnitude of the magnetic field, \( B \), in the gap.
(5) A spin-1/2 particle has a spin wavefunction, \( \chi = c \left( \begin{array}{c} 3 \\ 4i \end{array} \right) \). The two elements represent the spin up (top) and spin down (bottom) elements of the z-component of the spin.

a) Normalize this spin wavefunction by finding the value of \( c \).

b) If we measure \( S_z \), what is the probability of finding \( \frac{\hbar}{2} \)?

c) For a particle with this wavefunction, find the expectation value of \( <S_y> \).

Recall that \( S_y = \frac{\hbar}{2} \left( \begin{array}{cc} 0 & i \\ -i & 0 \end{array} \right) \).

d) For a particle with this wavefunction, what is the probability of measuring \( \frac{\hbar}{2} \) for \( S_x \)? Recall that \( S_x = \frac{\hbar}{2} \left( \begin{array}{cc} 0 & 1 \\ 1 & 0 \end{array} \right) \).
(6) Consider a two state system. In a certain basis, the Hamiltonian matrix is

\[ H = \begin{pmatrix} \varepsilon_1 & \lambda \\ \lambda & \varepsilon_2 \end{pmatrix} \]

where \( \lambda \) is a (real) parameter that can take any value from minus to plus infinity. Assume also \( \varepsilon_2 > \varepsilon_1 \).

a) Find the energy eigenvalues.

b) Study the limits \( \lambda \to 0 \) (expanding to quadratic order) and \( |\lambda| \to \infty \) (leading order).

c) Show that, because \( \varepsilon_1 \neq \varepsilon_2 \), the eigenvalues do not cross, that is, they can never be equal for any value of lambda.

d) Using the information in b) and c) make an approximate plot for the eigenvalues as a function of lambda.
7) Consider a particle trapped inside a box containing two regions, region A and region B. Region A has twice the volume of region B. When the particle is in region A, the system energy is U and when the particle is region B the energy is 2U. Assume that U > 0.

a) If the system is held at temperature T, derive the expression for the probability of finding the particle in region A.

b) What is the probability of finding the particle in region A at zero temperature?

c) What is the probability of finding the particle in region A at infinite temperature?
(8) Extreme relativistic particles have momenta \( p \) such that \( pc \gg Mc^2 \), where \( M \) is the rest mass of the particle. Show that the mean energy per particle of an extreme relativistic ideal gas is \( 3\tau \) if \( E \sim pc \), in contrast to \( 3/2\tau \) for the non-relativistic problem. Begin by determining the partition function for one particle. Note that \( \tau = k_B T \).
Useful Equations

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