Physics Graduate School Qualifying Examination

Fall 2010  Part I

Instructions: Work all problems. This is a closed book examination. Calculators may not be used. **Start each problem on a new pack of correspondingly numbered paper and use only one side of each sheet.** Place your 3-digit identification number in the upper right hand corner of each and every answer sheet. All sheets, which you receive, should be handed in, even if blank. Your 3-digit ID number is located on your envelope. All problems carry the same weight.

- Correct answers without adequate explanation/reasoning will not get full credit.
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- Use correct vector notation when appropriate.
- Your work must be legible and clear.

1. A hourglass (an ancient device for the measurement of time) sits on a scale. All the sand is initially in the upper bulb. The hourglass mass (not including the sand) is $M_H$ and the total sand mass is $m_s$. At $t=0$, sand starts to flow into the lower bulb at a constant rate of $\frac{dm}{dt}=\dot{\lambda}$. The sand falls through a height $L$. Find the scale reading as a function of time for all $t\geq 0$. 
2. A solid spherical ball (mass M, radius R) rolls without sliding down a wedge. The wedge is on a frictionless level table top, and the gravitational acceleration is given by g. The moment of inertia of the ball is

\[ I = \frac{2}{5} MR^2 \]

(a) Find the Lagrangian and Lagrange’s equations.

(b) Integrate the system’s equations of motion, to find the ball’s height as a function of time. The ball and the wedge are initially at rest and the ball’s center is initially H above the table surface.
3) Consider a coil of radius $a$ to which two small spheres of mass $m$ and charge $q$ are attached at a distance $b$, see the figure. The whole system is free to rotate around the vertical axis $z$. A current circulates through the coil, creating a magnetic field. The coil is long enough so that the magnetic field can be considered uniform inside (and along direction $z$) and zero at the position of the charges.

At $t=0$, the current starts to decrease in such a way that the intensity of the magnetic field inside the coil changes in time as $B = B_0 \left(1-t/t_0\right)$ until $t=t_0$ after which we have $B=0$.

(a) Argue that an electric field is generated and compute its direction and magnitude at the position of the charges. Use a quasistatic approximation, namely ignore radiation effects.

(b) Compute the torque exerted on the charges. Compute the total angular momentum of the system after the magnetic field is switched off ($t>t_0$)

(c) Discuss qualitatively whether or not the angular momentum is conserved between: (i) the initial state where the magnetic field is on and the charges are at rest, and (ii) the final state where the field is off ($B=0$) and the charges are rotating.
4) A common demonstration in low-temperature physics shows a magnet floating above a superconducting disk. To understand how it works consider a situation (as in the figure) where we model the magnet by a magnetic dipole of strength $\mu = (\mu_x, 0, \mu_z)$ situated at $x=0, y=0, z=a$ and the superconductor by a semi-infinite region of space, $z<0$ where the magnetic field vanishes $B=0$. The superconductor creates a surface current that screens all magnetic field inside. The boundary conditions on the surface are such that the normal magnetic field vanishes and the tangential one has a jump.

(a) Use symmetry arguments to show that the boundary conditions on the surface $z=\theta$ will be satisfied if one includes an image dipole of the same strength at the position $x=0, y=0, z=-a$. What should be the orientation of the image dipole? Verify your result by adding explicitly the magnetic field created by both dipoles at $z=\theta$.

(b) Using the previous result, compute the potential energy of the system. Show that it decreases with increasing $z$, implying that the force that the dipole experiences is repulsive, independently of its orientation.

If the dipole is oriented perpendicular to the surface, compute the surface current induced in the superconductor.

$$\vec{B} = 3 \frac{\vec{\mu} \cdot (\vec{r} - \vec{r}_0)}{|\vec{r} - \vec{r}_0|^5} (\vec{r} - \vec{r}_0) - \frac{1}{|\vec{r} - \vec{r}_0|^3} \vec{\mu}$$

$$U = -\frac{3(\vec{\mu}_1 \vec{r})(\vec{\mu}_2 \vec{r})}{r^5} + \frac{\vec{\mu}_1 \vec{\mu}_2}{r^3}$$
5) Consider an electron in a uniform magnetic field $\vec{B}$ pointing in the $z$ direction. Its Hamiltonian $H = -\gamma \vec{B} \cdot \vec{S}$ where $\gamma$ is the gyromagnetic ratio.

Given the Pauli matrices: $\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $\sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$, $\sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

a) Represent the Hamiltonian as a matrix in the representation in which $S_z$ is diagonal.

b) Find the eigenvalues and eigenstates of the Hamiltonian.

c) Write the time dependent Schrödinger equation for the electron spin state.

\[ i\hbar \frac{\partial \psi}{\partial t} = H \psi \]

d) At $t = 0$ the electron is in an eigenstate of $S_z$, with eigenvalue $\frac{+\hbar}{2}$. Find the expectation value of the $x$ component of the spin $\langle S_x \rangle$ as a function of time.
6) Consider a one dimensional quantum problem in which a beam of particles of mass \( m \) and energy \( E \) is incident from \( x = \) negative infinity, and moving in the positive \( x \) direction. The particles are scattered by potential \( V(x) \).

a) If \( V = \begin{cases} \ -V_0 & \text{for } x \leq 0 \\ \ -\infty & \text{for } x > 0 \end{cases} \)

Defining all terms, write the time-independent Schrodinger equation and give its general solution.

The rest of the problem considers the following potential, with \( a > 0 \):

\[
V = \begin{cases} 
0 & \text{for region I, } x < -a \\
-V_0 & \text{for region II, } -a < x \leq 0 \\
\infty & \text{for } x > 0
\end{cases}
\]

b) Give general solutions to the time independent Schrodinger equation for regions I and II separately, without matching them at \( x = -a \).

c) State the matching conditions on the wave functions at \( x = -a \).

d) What is the probability current in region II? How are the probability currents in regions I and II related? How does this relation affect the form of the solution in region II? Explain your answers.
7) Consider a system containing $N$ distinguishable particles. There are two single particle energy levels ($E_1$ and $E_2$) available to the particles, so that in general we may have $n_1$ particles, each with energy $E_1$, and $n_2$ particles, each with energy $E_2$, with $N = n_1 + n_2$. Suppose the system is in thermal equilibrium with a large thermal reservoir at temperature $T$. Derive the correct expression for the equilibrium value of $n_1/n_2$ by maximizing the total entropy of the system + reservoir, at fixed total energy.

Assume that $n_1 \gg 1$, and $n_2 \gg 1$, and make use of the Stirling formula

$$\ln(n!) \approx n \ln(n) - n.$$ 

a) for the system, derive an expression for

$$\delta S_{\text{system}} = \frac{\partial S_{\text{system}}}{\partial n_1} \delta n_1$$

b) for the reservoir derive an expression for

$$\delta S_{\text{res}} = \frac{\partial S_{\text{res}}}{\partial U} \frac{\partial U}{\partial n_1} \delta n_1$$

c) Combine these two results to derive the final result for $n_1/n_2$ as a function of $T$. 
8) A polar molecule has a permanent electric dipole moment of magnitude, $p$. Write down a general expression for the average macroscopic polarization (dipole moment per unit volume) for a dilute system of $n$ molecules per unit volume at temperature $T$ in a uniform electric field $E$. (Assume that the dipole moments can be treated classically.)
Useful Equations

\[ \vec{\nabla} \cdot \vec{E} = \frac{\rho}{\varepsilon_0} \quad \vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} \quad i\hbar \frac{\partial \Psi}{\partial t} = H\Psi \]

\[ \vec{\nabla} \cdot \vec{B} = 0 \quad \vec{\nabla} \times \vec{B} = \mu_0 \vec{J} + \mu_0 \varepsilon_0 \frac{\partial \vec{E}}{\partial t} \quad F = -kT \ln Z \quad Z = \sum_i e^{\frac{E_i}{kT}} \]

Lorentz Transformations Equations:

\[ x' = \gamma(x - vt) \quad t' = \gamma (t - \frac{v}{c^2} x) \quad y' = y \quad z' = z \quad \gamma = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}} \]

\[ E'_{x} = E_{x} \quad B'_{x} = B_{x} \]

\[ E'_{y} = \gamma (E_{y} - vB_{z}) \quad B'_{y} = \gamma (B_{y} + \frac{v}{c^2} E_{z}) \]

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1) A Super-Ball is a hard spherical rubber ball. For the purposes of this problem, assume that it undergoes perfectly elastic collisions and that there is no slip at the point of contact between the ball and the surface it encounters.

The moment of inertia of the ball about its center of mass is $\frac{2}{5}Ma^2$ where M is the mass of the Super-Ball and a is its radius.

A Super-Ball is dropped vertically onto a floor. It reaches the floor with velocity $v_y^0$ with an initial spin $\omega_z^0$. Find the final values for $v_y^f$, $v_z^f$, and $\omega_z^f$ in terms the given parameters. (The z-direction is into the plane of the paper.)
2) A wire of unstretched length, $l_0$, is extended by a distance $\beta l_0$ (where $\beta << 1$) when a certain mass $M$ is hung from its bottom end (first picture). If this same wire is connected between 2 points, A and B, that are a distance $l_0$ apart on the same horizontal level, and the same mass is hung from the midpoint of the wire (second picture), what is the depression, $y$, of the midpoint and what is the tension in the wire? [Assume that $y << l_0$ and that the mass of the wire is negligible.]
3. A particle with spin 1 is initially at rest and is polarized with its spin aligned with the +z axis. It subsequently decays into two electrons, which have spin 1/2.

(a) Working in a basis where the electron spins are quantized along the +z axis, write down all possible ways to couple the spins of the two electrons so as to obtain states with total angular momentum of 0 and 1. Assume that there is no relative orbital angular momentum.

(b) What is the probability for observing each of these possible two-particle final states?

(c) What is the probability of observing the final state with total angular momentum aligned along an axis ẑ' that is oriented at an angle θ with respect to ẑ?

Recall that the rotation operator in the basis of states with z quantized along the z axis can be expressed

\[ R^{(y)}_{m m'}(\theta) = \begin{pmatrix} \frac{1}{2}(1 + \cos \theta) & -\frac{1}{\sqrt{2}} \sin \theta & \frac{1}{2}(1 - \cos \theta) \\ \frac{1}{\sqrt{2}} \sin \theta & \cos \theta & -\frac{1}{\sqrt{2}} \sin \theta \\ \frac{1}{2}(1 - \cos \theta) & \frac{1}{\sqrt{2}} \sin \theta & \frac{1}{2}(1 + \cos \theta) \end{pmatrix} \]
4. Consider a particle confined to an infinite square well potential of the form

\[ V(x) = \begin{cases} 0 & \text{for } -a/2 < x < a/2 \\ \infty & \text{otherwise} \end{cases} \]

(a) Find expressions for the normalized eigenfunctions for the particle. Hint: \( \sin^2 \theta = \frac{1}{2} (1 - \cos 2\theta) \) and \( \cos^2 \theta = \frac{1}{2} (1 + \cos 2\theta) \).

(b) Find expressions for the corresponding energies of these states.

(c) Suppose a small perturbation of the form

\[ \delta V(x) = \begin{cases} \lambda & \text{for } x > 0 \\ 0 & \text{for } x < 0 \end{cases} \]

was introduced. Calculate the resulting changes in the energies of the states, to first order in \( \lambda \).
A thin uniform annulus of radius $R$, carrying total charge $Q$ and total mass $M$, rotates about its axis as shown above.

a) **Find the ratio of its magnetic dipole moment to its angular momentum.** This is called the gyromagnetic ratio.

b) **What is the gyromagnetic ratio for a uniform spinning sphere?** (This requires no new calculation; simply decompose the sphere into infinitesimal rings, and apply the result of part (a).)

c) **According to quantum mechanics, the internal angular momentum of a spinning electron is $1/2\hbar$, where $\hbar$ is Planck's constant.** What, then, is the electron's magnetic dipole moment? (This semi-classical result is off from the real value by a factor of almost exactly 2.)
A long coaxial cable carries current I (the current flows down the surface of the inner cylinder of radius a, and back along the outer cylinder of radius b) as shown in the figure above.

\[ a) \text{ Find the magnetic energy stored in a section of the cable of length } l. \]

\[ b) \text{ Calculate the self-inductance of the cable.} \]
7) What is the change of entropy that occurs when two moles of an ideal gas A (at temperature $T$ and pressure $p$) and three moles of an ideal gas B (at the same $T$ and $p$), are allowed to mix and come into equilibrium at the same $T$ and $p$? What if the gases are the same, e.g., $A$ and $A$?
8) A zipper has N links each of which can be in two possible states – closed with energy 0 and open with energy $\varepsilon$. The zipper can only open from the left, i.e. a link $j$ can be open only if all links to its left with indices $i$, $1 \leq i < j$ are open. The zipper is held at temperature $T$.

a. Find the partition function of this system.
b. Find the average number of open links.
Useful Equations

\[ \vec{V} \cdot \vec{E} = \frac{\nabla E}{\varepsilon_0} \quad \vec{V} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} \quad \frac{i\hbar}{\partial t} \frac{\partial \Psi}{\partial t} = H\Psi \]

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