

Quantum simulations of a freely rotating ring of ultracold and identical bosonic ions

F. Robicheaux* and K. Niffenegger

Department of Physics and Astronomy, Purdue University, West Lafayette, Indiana 47907, USA

(Received 14 April 2015; published 15 June 2015)

We report on two types of quantum calculations pertaining to a ring of ultracold and identical bosonic ions in a static, external magnetic field. The purpose of these calculations is to give examples of what could be expected in certain types of measurements of the rotation of this ring. The first type of calculation explores how well the ring reaches the rotational ground state when it starts in the ground state of a trapping potential that is then adiabatically switched off; the trapping potential is designed to frustrate the rotation and raise the energy scale to the point where standard cooling techniques could reach the ground state. The second type of calculation shows what kind of rotation signal might be measured using two types of measuring schemes. The focus is on how the signal changes with the temperature of the ring and with how the measurement is performed.

DOI: [10.1103/PhysRevA.91.063618](https://doi.org/10.1103/PhysRevA.91.063618)

PACS number(s): 03.75.Hh, 52.55.Dy, 34.80.Lx, 52.65.—y

I. INTRODUCTION

There have been several recent papers discussing whether a closed quantum system could exhibit the features of a time crystal. The basic idea of time crystal was discussed in Refs. [1,2] and revolves around the question of whether “time-translation symmetry might be spontaneously broken in a closed quantum-mechanical system.” The interest in such a question arises in analogy with spontaneous symmetry breaking in different physical systems. In spontaneous symmetry breaking, the equations that govern a system exhibit a type of symmetry that is not exhibited by the lowest-energy states of that system. An example is a crystal: the Hamiltonian is invariant under any translation but the crystal ground state is only invariant under translation by a multiple of a lattice vector. Reference [1] claimed that, similar to the spontaneous symmetry breaking in a crystal, there are closed quantum systems that exhibit spontaneous symmetry breaking with respect to time translation.

Reference [3] proposed a specific example of a time crystal where the physical system was a ring of cold, trapped ions with an external magnetic field normal to the plane containing the ring. They noted that the ground state of this system can have the ion ring rotating at a nonzero frequency that depends on the strength of the external magnetic field and parameters of the trap (e.g., trap radius, number of ions, etc.). They suggested that this system might be more easily realized in an experimental setting than the examples in Ref. [1]. Before discussing some of the properties of this system, we note that Ref. [1] was the subject of a critical comment [4] and a response [5] as well as a more in-depth critique [6]. The critiques question the existence of time crystals, while the responses argue for their existence. We do not directly address this controversy. In this paper, we address the simpler question of how a ring of identical ions responds to specific manipulations. Since this system was suggested by Ref. [3] as an example, the solutions of Schrodinger’s equations and the numerical simulations we present will address some of the quantum properties of this system. Thus some of our calculations may be relevant to the Comment [7] (and response

to the Comment [8]) to Ref. [3] which addressed the feasibility of the proposed measurements.

In all of our calculations, we solve the time-dependent Schrodinger equation for the ions to simulate the response of the ion ring. In these calculations, we assume that technical issues regarding a “freely rotating ion ring crystal” has been solved as discussed in Ref. [9]. A ring with approximately 400 Ca^+ ions has been demonstrated in Ref. [10]. In the proposed experiments, a Paul trap [11–13] for the ions will use an oscillating electric field to trap ions in a ring a few 100 μm above the surface of the electrodes [14–16]. The geometry is different from the more typical linear traps [12,17], but if the temperature of the ions is reduced enough they will form a crystal (as has been seen in many linear traps [17]) with equal spacing between ions. The radio frequency electric field causes a small oscillation of the ions perpendicular to the direction along the ring. The oscillation leads to a ponderomotive force that pushes the ion towards the spatial positions where the magnitude of the electric field is a minimum. Taking z to be perpendicular to the ring, the oscillation provides an effective potential well (i.e., a pseudopotential) in z and r centered at some height z_0 and radius R . For the time crystal, the ions need to be cooled past the point of crystal formation to temperatures where the vibrational motion of the ions are in the ground state; this is an experimentally challenging regime. See Ref. [9] for more details of the proposed Paul trap.

As noted in Ref. [9], a ring of identical ions are, perhaps unexpectedly, less prone to pinning as the number of ions increases. This is because the ion ring better simulates a continuous charge with more ions. By assuming the experimental difficulties of the ion ring are solved, this implies that the ions can be treated as being in a nearly perfect circle and that there are no uncontrolled external fields that pin the ring. It also assumes the isotopic purity of the ions and that they can be placed in the same atomic state (for example, using optical pumping). In all examples below, we choose the ions to be bosons so that the wave function is symmetric upon interchange. Using identical fermions does not change the qualitative results.

The experiment proposed in Ref. [3] is that of a freely rotating ring of N identical ions that has a magnetic field through it. For nonidentical ions, the spacing of the energy levels decreases as the number of ions increases because the

*robichf@purdue.edu

moment of inertia of the ring increases. If identical bosons are in identical internal states, the symmetrization means the wave function should repeat every $2\pi/N$ radians. Thus the allowed angular momenta for an ion is not $m\hbar$ but $Nm\hbar$ where m is an integer. The quantized energies of rotation with no magnetic field have the form $E_m = \hbar^2 N^2 m^2 / (2NMR^2) = NE_{sc}m^2$ with $E_{sc} = \hbar^2 / (2MR^2)$. This shows that the spacing of the energy levels increases as the number of ions in the ring increases which is opposite to what would happen if even one of the ions were distinguishable.

In Sec. II, we will repeat the arguments for treating the ions as a rigid ring. The results for a rigid ring of ions has been presented in many places. The Hamiltonian for a rigid ring of ions with a magnetic field perpendicular to the ring can be written as

$$H = \frac{E_{sc}}{N} \left(\frac{1}{i} \frac{\partial}{\partial \Theta} - Na_0 \right)^2, \quad (1)$$

where the $0 \leq \Theta \leq 2\pi$ can be considered a position on the ring, $E_{sc} = \hbar^2 / (2MR^2)$, N is the number of ions, and $a_0 = \pi R^2 B / (h/e)$ is the scaled magnetic flux. To be specific, we can take the position of the j th ion to be at $\theta_j = \Theta + j2\pi/N$. Instead of considering Θ to be a position on the ring, the Θ can also be derived from the normal mode coordinates in Ref. [18] and is their coordinate $u - \pi$ [see their Eq. (17)] which is simply the phonon mode related to $k = 0$. If every ion is an identical boson in the same internal state, then the wave function must have the property of repeating when $\Theta \rightarrow \Theta + (2\pi/N)$. This means the allowed energy states have $0, \pm N, \pm 2N, \dots$ rotational quanta. This gives an energy spectra

$$E_m^{\text{ident}} = E_{sc} N (m - a_0)^2, \quad (2)$$

where $m = 0, \pm 1, \pm 2, \dots$. If one of the ions is different in any way, then the wave function has the property of repeating when $\Theta \rightarrow \Theta + 2\pi$. The allowed energy states now have $0, \pm 1, \pm 2, \dots$ rotational quanta which gives an energy spectra

$$E_m = E_{sc} N^{-1} (m - Na_0)^2, \quad (3)$$

where $m = 0, \pm 1, \pm 2, \dots$. The E_m^{ident} of Eq. (2) is contained in Eq. (3) by putting on the restriction that $m = 0, \pm N, \pm 2N, \dots$

Although the identical boson case has energy spacing N^2 greater than the distinguishable case, the spacing of the rotational energy levels is much smaller than typical temperatures that can be reached for ions. Section III discusses one possible method for getting the system to lower energies; this method is based on a suggestion made to us by Li [19] to use the ponderomotive shift from a standing light wave. Reference [9] discussed a different method based on a strong electric field.

Reference [3] proposed preparing the ring of identical bosonic ions into an eigenstate with energy given by Eq. (2). To measure the rotation, an ion would be marked by (for example) changing the internal state of one ion or a group of ions. This would change the state from an eigenstate, where observables do not have time dependence, to a state represented by a superposition of eigenstates with energies given by Eq. (3). This superposition is a wave packet which evolves in time. In Sec. IV, we discuss two methods to mark a ring and give examples of the resulting time-dependent signal.

II. JUSTIFICATION FOR USING A RIGID RING

The ideas in this section have been expressed in many papers (for two, see Refs. [3,9]). This section is included to give the basic idea of what physical parameters are needed. In the calculations, we use ${}^9\text{Be}^+$ as the bosonic ion. Estimates can be made for other ions by scaling the energies appropriately with the ion mass.

There are several possible ways that the ring of ions might not be well represented as a rigid ring. Many of these are under experimental control. For example, the oscillation frequency of ions perpendicular to the ring plane needs to be high enough that the ions are in the lowest vibrational state. If this is not the case, then the transverse phonons can become a mechanism for ‘‘marking’’ the ring. There are also transverse phonons in the plane of the ring which also need to be in the lowest mode. These frequencies are determined by trap parameters and are, thus, beyond the scope of our studies. By making the trapping potential ‘‘tighter,’’ these frequencies can be raised. How tight the trapping potential can be is an experimental issue. There are also various ring instabilities which must be avoided that occur when too many ions are in a given trap; many of these issues are discussed in Ref. [9].

One of the intrinsic modes that must be considered are the longitudinal phonons of the ring. These were discussed in detail in Ref. [18] and only depend on the ion charge and mass and the ring radius when the transverse modes can be ignored. Using the expressions from this paper one can show that the energies of the different longitudinal phonon modes are given by

$$\begin{aligned} \varepsilon_{v_j, j} &= E_{sc} \sqrt{\frac{2q^2}{4\pi\epsilon_0 R E_{sc}}} v_j \omega_j \\ \omega_j^2 &= \sum_{n=1}^{N-1} \frac{1 + \cos^2(n\pi/N)}{4 \sin^3(n\pi/N)} \sin^2(jn\pi/N), \end{aligned} \quad (4)$$

where $E_{sc} = \hbar^2 / (2MR^2)$ is the scale of the rotational energy, M is the mass of a single ion, R is the radius of the ring, q is the charge of the ion, N is the number of ions, v_j is the number of quanta, and ω_j is the scaled angular frequency of the j th phonon. The lowest frequency phonon is for $j = 1$. For 100 ions, $\omega_1 \simeq 11.6$ and $\omega_2 \simeq 21.3$, while for 90 ions $\omega_1 \simeq 10.9$ and $\omega_2 \simeq 19.9$. For a ring of singly charged ions with a radius of $60 \mu\text{m}$, the $2q^2 / (4\pi\epsilon_0 R) \simeq 3.84 \times 10^{-24} \text{ J} \simeq k_B 0.278 \text{ K}$. For an ion of nine atomic mass units, $E_{sc} = 1.03 \times 10^{-33} \text{ J} \simeq k_B 74.8 \text{ pK}$. Combining these gives an energy spacing for the lowest frequency phonon of $\sim k_B 50 \mu\text{K}$. Compared to the energy spacing of the rotational motion, $NE_{sc} \sim k_B 7.5 \text{ nK}$, this phonon energy is four orders of magnitude higher and will be frozen out in an experiment. Using the lowest frequency phonon mode, we can use the length scale of the ground state of a harmonic oscillator to estimate the ions are localized to their position in the ring with a spatial width with a scale less than a few 10^3 nm , whereas the ion separation is a few μm . Thus the ion positions are at their equilibrium position with an accuracy of approximately 1%.

Using the discussion in Sec. III of Ref. [9] as a guide, we chose to perform all of our calculations with a radius of $60 \mu\text{m}$, 90 ions, and mass of 9 amu.

III. ADIABATIC COOLING USING PONDEROMOTIVE SHIFT

One of the difficulties with the experiment is the very small energy scale of the rotational energies of the ring which is of order nK. We will assume that the ions can be cooled to the several μK scale, which is somewhat ambitious but has been achieved [20,21]. If the ions can be cooled to this level one can pursue schemes where the cooling takes place with a potential that pins the ring and raises its energy scale. Once the ring reaches the ground state of the pinning potential, this potential can, in principle, be adiabatically switched off [22] so that the ring will end in the ground state of the freely rotating ring. Reference [9] proposed using a uniform electric field to push all of the ions to one side of the circular trap. We will investigate a different situation where the ions experience a pinning potential from the ponderomotive force from a standing light wave.

The potential energy of the ring from the standing light wave is proportional to the sum of the intensity of the light at the position of each ion. We will assume that the light intensity is only modulated in the x direction and has the form

$$I(x) = I_0 \cos^2(kx), \quad (5)$$

where k is the wave number in the x direction. Taking the position of the j th ion to be $\theta_j = \Theta + 2\pi j/N$, the potential energy will be proportional to

$$V_{\text{pot}} \propto -V = -\sum_j \cos^2(kx_j), \quad (6)$$

where $x_j = R \cos(\Theta + 2\pi j/N)$. This dimensionless quantity must be in the range $0 \leq V \leq N$. Furthermore, this potential repeats when $\Theta \rightarrow \Theta + 2\pi/N$. In the discussion below, it is important to remember that \cos^2 repeats every π radians.

Figure 1 shows a schematic of how the potential arises for a ring with an even number of ions and a ring with an odd number of ions. The lines mark the positions of the intensity maximum and the asterisks mark the positions of the ions for one possible ring orientation. For an even number of ions, the intensity maximum from the light can line up with atoms on both the top and bottom of the ring when the separation of the intensity maxima is approximately $2\pi R/N$. This suggests one possible good choice is $k \simeq (1/2)N/R$. For an odd number of

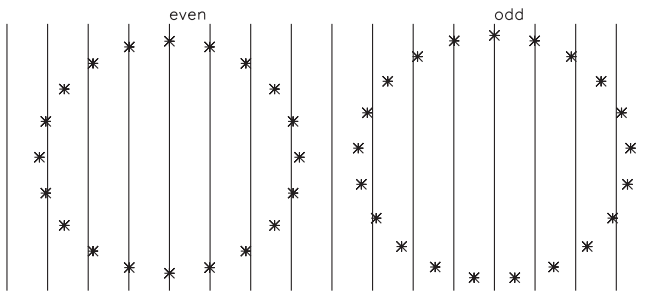


FIG. 1. Schematic of how a standing light wave can pin the ion ring. The lines represent the maxima of the light intensity for the standing wave. The ions are attracted to the light maxima. See text for discussion of the qualitative difference between even and odd numbers of ions.

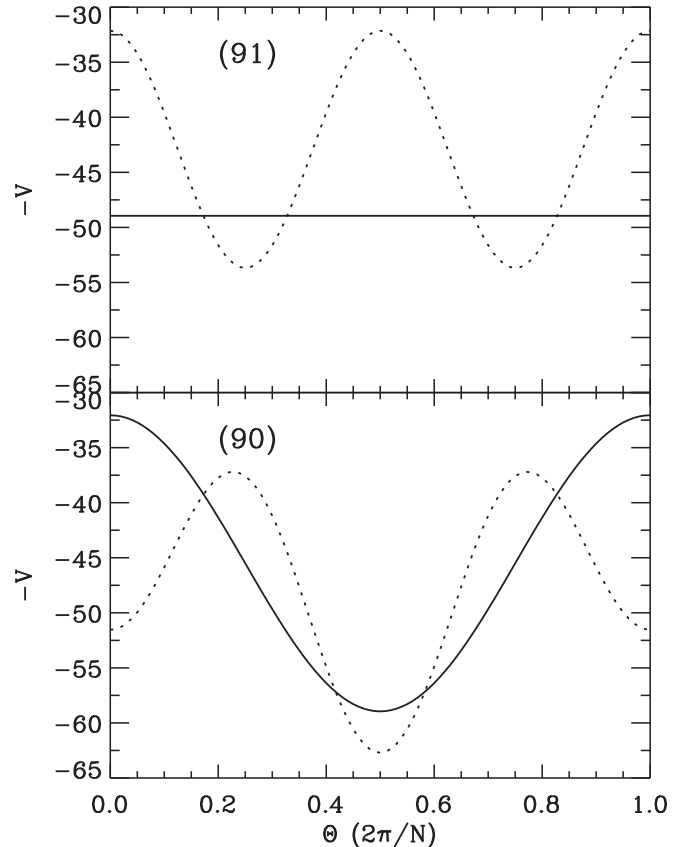


FIG. 2. Calculation of the $-V$ from Eq. (6) for 90 and 91 ions. The solid line is for $k = 0.52N/R$. The dotted line is for $k = 1.014N/R$ for $N = 90$ and $1.024N/R$ for $N = 91$. These values for k were chosen to maximize the well depths. The potential energy is typically proportional to $-V$ for red detuned light.

ions, the atoms line up with maxima on the top part of the ring but are at minima on the bottom part of the ring. Thus, for intensity maxima separated by $\simeq 2\pi R/N$, the odd number of atoms has almost no Θ dependence for the potential. If the k is chosen to be approximately $2\times$ larger, then the ions can be aligned with the maxima for both even and odd numbers of ions. This suggests that another possible good choice is $k \simeq N/R$. However, the larger k gives two potential minima which has a possibly terrible effect on adiabatic cooling as will be discussed below.

Figure 2 shows the calculation of $-V$ from Eq. (6) for a 90 and a 91 ion ring. The solid line is for the case when $k \simeq (1/2)N/R$. The $N = 90$ case has a single minimum in the potential, while the $N = 91$ case almost no variation with Θ . The $\max(V) - \min(V)$ for the $N = 90$ case is $\simeq 27$. This means the effect of the potential is $27\times$ larger than that from one ion. However, this needs to be counterbalanced by the mass of the ion ring being $90\times$ larger. Thus the oscillation frequency in the well is approximately $\sqrt{27/90} \simeq 0.55$ times that for a single ion in the well.

The dotted lines are for $k \simeq N/R$ with the exact values given in the caption. For this case, both the even and the odd number of ions shows a potential variation. For the $N = 90$, the $\max(V) - \min(V)$ is $\simeq 25$, while it is $\simeq 22$ for the $N = 91$ case. Since the well depth is similar to that for the

smaller k , one is tempted to think it's better to use the larger k because the frequency is proportional to the square root of the curvature of $-V$ near the minimum since the time needed for adiabatically switching off the potential scales like the inverse of the frequency. However, the cases with $k \simeq N/R$ give double wells. For an odd number of ions, the wells are equally deep, while for the even case the potential well at $\Theta = (1/2)(2\pi/N)$ is clearly deeper. For odd N , this will prevent adiabatic cooling to the ground rotational state because the ground and first excited state are equally populated. When the adiabatic cooling step occurs, the ring would be left in an equal mixture of ground and first excited states. For the even case, there could be a benefit if the cooling can be done so that the ring ends in the deeper well most of the time. We will focus on the case where $k \simeq (1/2)N/R$ for our numerical studies as that will most simply illustrate the features of adiabatic cooling. In the next section, we will see that the behavior of a ring in the ground and in the first excited state is distinguishable so it might be worthwhile to deliberately have a mixture of the lowest two states.

From Fig. 2, we can see that the ring as a whole will have a pinned frequency comparable to that for a single ion. We will assume that the one ion well depth is sufficient to reach the ground vibrational state which means we will assume the ring can also be cooled to the ground state. The main question is how fast can the laser be turned off so that the ion ring will finish in the ground rotational state. To address this, we solved the time-dependent Schrödinger equation for the ring:

$$i\hbar \frac{\partial \Psi(\Theta, t)}{\partial t} = \frac{E_{sc}}{N} \left(\frac{1}{i} \frac{\partial}{\partial \Theta} - Na_0 \right)^2 \Psi(\Theta, t) - \epsilon(t)[V(\Theta) - V_{\max}] \Psi(\Theta, t), \quad (7)$$

where the boundary condition is $\Psi(\Theta + 2\pi/N, t) = \Psi(\Theta, t)$, $\epsilon(t)$ has units of energy and smoothly turns off, and V_{\max} is the maximum value of $V(\Theta)$. By having the potential proportional to $-(V - V_{\max})$ instead of just $-V$ we have set the minimum of the total potential energy to zero.

We solved for the time-dependent wave function by discretizing the Hamiltonian using a grid of points in Θ between zero and $2\pi/N$ with a periodic boundary condition. The resulting matrix representation of H is tridiagonal except for nonzero elements in the upper-right and lower-left corners. The time propagation was carried out using the implicit Crank-Nicolson algorithm

$$\Psi(t + \delta t/2) = \frac{1 - iH(t)\delta t/2}{1 + iH(t)\delta t/2} \Psi(t - \delta t/2), \quad (8)$$

which has a one step error of order δt^3 . We used the numerical eigenstate of H for $\epsilon(0)$ to get the initial ground-state wave function.

One of the difficulties in trying to obtain the most population in the ground state is what functional form to choose for $\epsilon(t)$, i.e., how to have $\epsilon(t) \rightarrow 0$. The difficulty is seen in Fig. 3 which plots the eigenvalues of the Hamiltonian. When ϵ is large, the spacing of energy levels is large and the ϵ can be decreased quickly. As the ϵ decreases, the spacing gets smaller and the rate that ϵ changes should also decrease. This means most of the decrease in ϵ can occur very quickly but the final decrease towards zero will happen slowly and will be of the

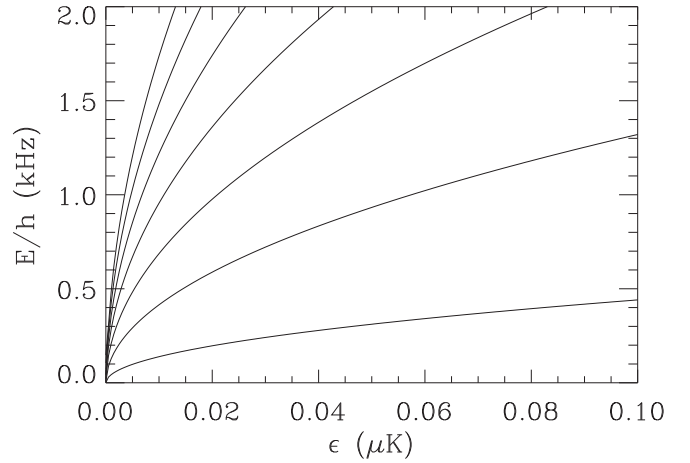


FIG. 3. Lowest seven eigenvalues (in units of kHz) of the Hamiltonian in Eq. (7) for different values of ϵ (in units of μK). The calculations are performed for $a_0 = 1/4$, $R = 60 \mu\text{m}$, and a mass of 9 amu.

order of $\hbar/\Delta E$ evaluated at $\epsilon = 0$. For the parameters in Fig. 3, the difference between the ground state and first excited-state energy is approximated by

$$\Delta E(Hz) = \sqrt{49 + 1.5 \times 10^{11} \epsilon^2 / [1 + 2.0 \times 10^4 \epsilon]}, \quad (9)$$

where ϵ is in μK ; this expression is accurate to approximately 10% over the full range of ϵ . At small ϵ , the $\Delta E \rightarrow 7 \text{ Hz}$. At large ϵ , the ΔE is proportional to $\sqrt{\epsilon}$ because the potential is well approximated by a harmonic oscillator when the ring is strongly confined near the potential minimum.

In our propagation, we used a simple algorithm that numerically decreased ϵ using the form

$$\epsilon(t + \delta t) = \epsilon(t) \exp(-\beta(t)\delta t \Delta E), \quad (10)$$

where ΔE is in Hz and $\beta(t)$ was a positive dimensionless number that starts at zero and then smoothly increases. Larger β is equated with a faster turnover. We chose the form $\beta(t) = \beta_\infty / (1 + \exp[-10 + t \Delta E_0/2])$, which smoothly turns on over a time of approximately $2/\Delta E(Hz)$. The final results depended weakly on the specific forms we picked for how we turned off the laser.

Figure 4 shows the population in the four lowest-energy states as a function of the turn-off time. The turn-off time was defined to be the time when $\epsilon(t)$ first dropped below 10 pK. We started $\epsilon(0) = 10 \mu\text{K}$. The time scale is increased by a factor of approximately 2 if the turn-off time is defined to be when the $\epsilon(t)$ first dropped below 1 pK. We only plot the range where the population in the ground state is larger than $\sim 60\%$. As is typical with time varying Hamiltonians, the population in the initial state stays near 1 until the time scale of the variation becomes shorter than $\sim \hbar/\Delta E$. The time scale associated with $\hbar/\Delta E$ when $\epsilon = 0$ is approximately 140 ms. Thus it would be surprising if the potential can be turned off substantially faster than that in Fig. 4 and still have most of the population in the initial state. Thus it appears that the relevant time scale for this type of pinning potential is a few 10's ms and is mainly determined by the rotational energy levels as the pinning is turned off.

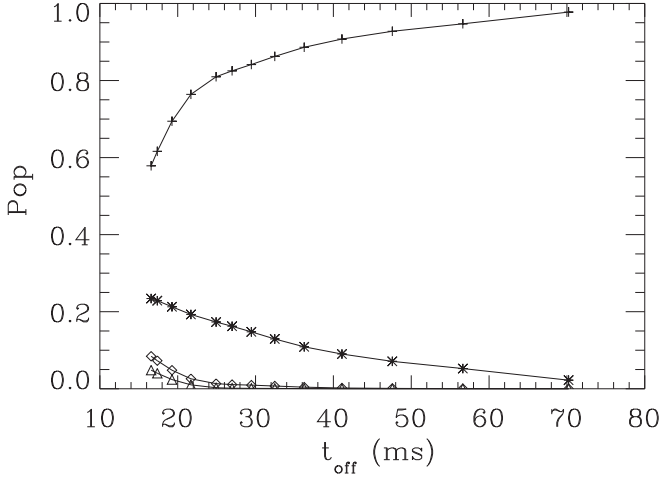


FIG. 4. Population in the ground (+), first excited (*), second excited (diamonds), and third excited (triangles) states as a function of the turn-off time.

This time scale may be too slow for an experimental implementation. The ions have a time scale over which they are heated by external noise and other processes. If the turn-off time is too long, the ions will be heated out of the ground state during the adiabatic cooling step. Since all pinning potentials have the same limit as the potential is turned off, it seems probable that all pinning potentials will have a turn-off time of a few 10's ms. This time scale is much shorter than the rotational period of a marked ring which determines the duration of a rotation experiment. Thus the main difficulty will be in having the ring reheat between the end of the adiabatic cooling and the start of the measurement of the ring rotation.

IV. SIGNAL FROM A MARKED RING

Reference [3] described a generic experiment where the ion ring is marked in order to experimentally determine whether it is rotating and in what direction. To simulate this process, a theory for this many-body system under the action of a one-body interaction needs to be developed. There are many possible ways of marking an ion. We will treat the situation where internal quantum numbers are changed which leaves the interaction of the ion with external potentials unchanged. To simplify the treatment, we will assume the marking occurs within an effective two-state system.

All of the ions are initially in state $|a\rangle$ and the marking process causes $|a\rangle \rightarrow |a\rangle C_a + |b\rangle C_b$ with $|C_a|^2 + |C_b|^2 = 1$. The transition must be localized to a region of order $\delta\theta \ll 2\pi$ in order for the ring to be marked. The number of ions in this region is approximately $N\delta\theta/(2\pi)$ and will be assumed to be much larger than 1. In the discussion of this process in Ref. [3], it was assumed the number of marked ions is approximately 1 which means that the probability for any one atom to be excited is small. We will choose the form of the interaction to give

$$|a\rangle \rightarrow e^{i\alpha(\theta)\sigma_x} |a\rangle = |a\rangle \cos(\alpha(\theta)) + |b\rangle i \sin(\alpha(\theta)), \quad (11)$$

where $\sin^2(\alpha(\theta))$ is the probability to mark one ion as a function of the angular position, θ , of the ion. The probability of exciting

any one ion is less than the inverse of the number of ions in the transition region; this implies $\alpha^2 < 2\pi/(N\delta\theta)$ which means the α is typically much less than 1. We will define η as the angle where α is a maximum with a value $\alpha_0 \equiv \alpha(\eta)$ and where $\delta\theta$ is the width of the marking in angle. This assumes the marking is done using a Hamiltonian operation and not a method that requires dissipation like a spontaneous photon emission. A possible method could be a stimulated Raman transition that changes the hyperfine state of the ion; if the two laser beams are oppositely directed, the momentum kick from the photon absorption then emission will be too small to excite phonons.

The initial wave function for the ring can be written as

$$\Psi_0 = \frac{1}{\sqrt{2\pi}} e^{im\Theta} |aaa \dots a\rangle \times \frac{\psi(1,2,3, \dots N) + \psi(2,1,3, \dots N) + \dots}{\sqrt{N!}}, \quad (12)$$

where $m = 0$ or $\pm N$ or $\pm 2N$, etc., and the $\psi(1,2,3, \dots N)$ is a spatial wave function. The spatial wave function is such that the ion coordinate in the n th place is localized at $\Theta + n \times 2\pi/N$. Thus the $\psi(2,1,3, \dots N)$ has ion 2 localized at $\Theta + 2\pi/N$, ion 1 localized at $\Theta + 4\pi/N$, and ions $n \geq 3$ localized at $\Theta + n2\pi/N$. The spatial wave functions have the property where all ions need to be lined up for the overlap to be nonzero. For example,

$$\begin{aligned} \langle \psi(1,2,3, \dots N) | \psi(1,2,3, \dots N) \rangle &= 1, \\ \langle \psi(2,1,3, \dots N) | \psi(1,2,3, \dots N) \rangle &= 0. \end{aligned} \quad (13)$$

This is because the ion localization length scale is approximately a few 10's nm compared to the ion separation of a few μm . The discussion of these length scales is in Sec. II.

The wave function just after the step that marks the ion has the form

$$\Psi_1 = \prod_{n=1}^N [1_{-n} \cos(\alpha(\theta_n)) + i \sigma_{x,n} \sin(\alpha(\theta_n))] \Psi_0, \quad (14)$$

where $\theta_n = \bar{\theta}_n + \Theta$ with $\bar{\theta}_n$ the average position of the n th ion in the ring which will depend on the ordering in $\psi(\dots)$. This function is not an eigenstate and will have nontrivial time dependence.

Because the initial wave function is symmetric and the marking is also symmetric, all of the physically relevant quantities can be obtained from the part of the wave function multiplying the $\psi(1,2,3, \dots N)$ term. In what follows, we will only discuss this part of the wave function. The full wave function can be found by simple exchanges of the ion indices.

A. Weak marking

The experiment will probably work in a regime where approximately one atom is marked, but this is difficult to treat theoretically. We will first discuss the case where the average number of marked atoms is much less than one since it will allow a simplified treatment and will contain much of the relevant physics.

As discussed in the previous section, we only need to solve for the part of the wave function multiplying the

$\psi(1,2,3, \dots N)$ ordering. The rotation angle α is always much less than 1 for the case of weak marking. Thus the product in Eq. (14) reduces to a sum

$$\Psi_1 \simeq \Psi_0 + i \sum_{n=1}^N \alpha(\theta_n) \sigma_{x,n} \Psi_0, \quad (15)$$

which is not an eigenstate. We will use the superposition principle to find the wave function after the marking. For a general Hamiltonian with eigenstates $y_n(x)$ and energies E_n , if you multiply an eigenstate at time $t = t_0$ with the function $f(x)$, $\Psi(x, t_0) = f(x)y_n(x) \exp(-iE_n t_0/\hbar)$, the result is generally not an eigenstate and has the form for $t \geq t_0$

$$\Psi(x, t) = e^{-iE_n t/\hbar} \sum_{n'} y_{n'}(x) e^{-i(E_{n'} - E_n)(t-t_0)} f_{n',n}, \quad (16)$$

where $f_{n'n} = \langle y_{n'} | f(x) | y_n \rangle$.

If the marking is done at time t_0 , the part of the wave function multiplying the $\psi(1,2,3, \dots N)$ can be written as

$$\tilde{\Psi}_1 = \left[1 + i \sum_{n=1}^N F_n(\Theta, t) \sigma_{x,n} \right] |aa \dots a\rangle \Phi_0(\Theta, t), \quad (17)$$

with

$$\Phi_0(\Theta, t) = e^{-iE_m t/\hbar} \frac{1}{\sqrt{2\pi}} e^{im\Theta}, \quad (18)$$

where the E_m is defined in Eq. (3), $m = 0$ or $\pm N$ etc., and the function $F_n(\Theta, t)$ is defined by

$$F_n(\Theta, t) = \sum_{m'} e^{i[\delta m' \Theta - (E_{m'} - E_m)(t-t_0)/\hbar]} \alpha_{m',m}^{(n)}, \quad (19)$$

where $\delta m' = m' - m$ and the $m' = 0, \pm 1, \pm 2, \dots$. The expression for $F_n(\Theta, t)$ contains $\exp(i\delta m' \Theta)$ instead of $\exp(im' \Theta)$ because it multiplies the Φ_0 ; the $\exp(-im\Theta)$ cancels the $\exp(im\Theta)$ in the Φ_0 . The matrix

$$\begin{aligned} \alpha_{m',m}^{(n)} &= \langle m' | \alpha(\theta_n) | m \rangle \\ &= \frac{1}{2\pi} \int_0^{2\pi} d\Theta e^{-i\delta m' \Theta} \alpha(\bar{\theta}_n + \Theta) \\ &= e^{i\delta m'(\bar{\theta}_n - \eta)} \frac{1}{2\pi} \int_{-\pi}^{\pi} d\tilde{\Theta} e^{-i\delta m' \tilde{\Theta}} \alpha(\eta + \tilde{\Theta}) \\ &= \frac{\alpha_0 \delta \theta}{\sqrt{8\pi \ln 2}} e^{i\delta m'(\bar{\theta}_n - \eta)} e^{-\delta m'^2 \delta \theta^2 / (8 \ln 2)}, \end{aligned} \quad (20)$$

where $\bar{\theta}_n = n2\pi/N$ and the last step assumed $\alpha = \alpha_0 \exp(-2 \ln(2)[\theta - \eta]^2 / \delta \theta^2)$ in the neighborhood of η . One can show that at $t = t_0$ the $|F_n(\Theta, t_0)|$ is peaked at the value defined by $\Theta + \bar{\theta}_n - \eta = 0$, which is when the n th ion is at the peak of α .

There are several possible ways of detecting the mark. One example is to do a pump-pump type experiment. For example, at $t = 0$, the ring can be marked at $\theta = \eta$. At a later time $t = t_f$, the marking operation can be performed again but at the angle $\theta = \eta_f$. The two amplitudes for excitation can add constructively or destructively if the ring has rotated by the angle $\eta_f - \eta$. By varying the phase of the second marking, one can detect whether or not the ring has rotated. We have performed calculations of this process and found that it could

detect the rotation. We will present the results for a simpler, pump-probe method for detecting the rotation. We assume that the detector can only detect ions in state $|b\rangle$ and ignores those in state $|a\rangle$.

We will define the probability for detecting an ion in state $|b\rangle$ at angle θ to be $\gamma(\theta - \eta_d)$, where η_d is the peak of the detection efficiency. Using the fact that ion n is at the position $\Theta + \bar{\theta}_n$, the probability for detecting an ion at time t is

$$P = \sum_{n=1}^N \int_0^{2\pi} d\Theta \gamma(\Theta + \bar{\theta}_n - \eta_d) |F_n(\Theta, t)|^2, \quad (21)$$

where the $\bar{\theta}_n = n2\pi/N$. One can show from the expressions above that $F_n(\Theta - \bar{\theta}_n, t)$ is independent of n . Thus the probability is

$$P(\eta_d) = N \int_0^{2\pi} d\Theta \gamma(\Theta - \eta_d) |F_n(\Theta - \bar{\theta}_n, t)|^2 \quad (22)$$

for $1 \leq n \leq N$. If the detection scheme is sharply peaked in angle, the $P(\eta_d) \propto |F_n(\eta_d - \bar{\theta}_n, t)|^2$. If the spreading of the wave packet is less than $\sim \pi$, the summation in m can be done analytically, which will give

$$P(\eta_d, t) = |\alpha_0|^2 \frac{\Delta\theta(0)}{\Delta\theta(t)} \exp\left(-\frac{1}{2} \frac{[\eta_d - \eta_m(t)]^2}{\Delta\theta^2(t)}\right), \quad (23)$$

where $\delta\theta$ is the FWHM at $t = t_0$, $\Delta\theta^2(0) = \delta\theta^2 / (8 \ln 2)$, $\Delta\theta^2(t) = \Delta\theta^2(0) + \omega_0^2(t - t_0)^2 / (2\Delta\theta(0))^2$, $\omega_0 = 2E_{sc} / (N\hbar)$, and $\eta_m(t) = \omega_0(m - Na_0)(t - t_0)$. This approximation shows that the packet behaves like a spreading Gaussian for short times with the peak having an angular frequency equal to $\omega_0(m - Na_0)$.

Figure 5 shows the results of calculations for the same parameters of Fig. 3 with $\delta\theta = \pi/8$. The different lines show different detection times separated by 1/2 s intervals. The ring

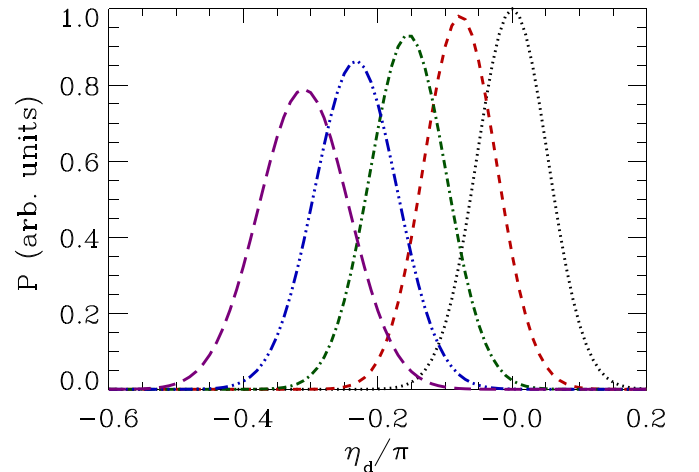


FIG. 5. (Color online) Angular distribution of the marked ion as a function of angle for times of 0 s (dotted), 1/2 s (dashed), 1 s (dash-dot), 3/2 s (dash-dot-dot-dot), and 2 s (long dash) after the marking occurs. All of the P 's have been scaled by the same factor so that the maximum at $t = 0$ is approximately 1. The calculations are performed for $a_0 = 1/4$, $R = 60 \mu\text{m}$, and a mass of 9 amu. Using the approximation from Eq. (23) gives results indistinguishable from those plotted here.

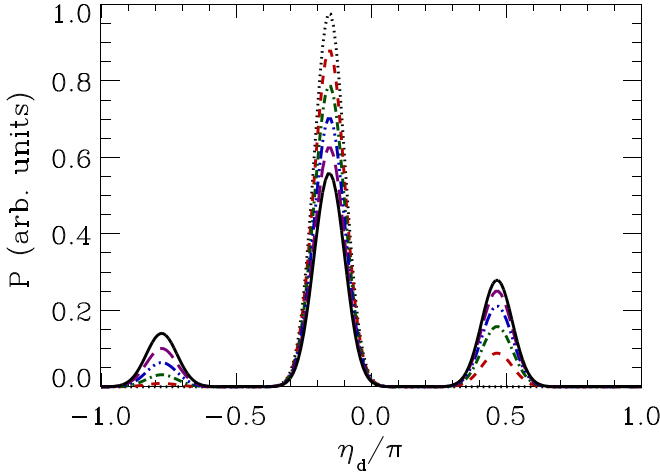


FIG. 6. (Color online) Angular distribution of the marked ion as a function of angle at the time 1 s for different amounts of contamination from the first and second excited states. The population in the first excited state was f times that in the ground state and the population in the second excited state was f^2 times the ground state. The calculations are $f = 0$ (dotted), $f = 0.1$ (dashed), $f = 0.2$ (dot-dash), $f = 0.3$ (dash-dot-dot-dot), $f = 0.4$ (long dash), and $f = 0.5$ (solid).

is marked at $\theta_0 = 0$. The marked ion rotates in the negative Θ direction because a_0 was chosen to be positive and the ring is assumed to start in the ground state: $m = 0$. The decrease in height of the $P(\eta_d)$ is due to the spreading of the wave packet due to dispersion. We followed the evolution of the wave packet to much longer times, 50 s, and found that the packet dispersed to approximately cover the full 2π at approximately 25 s which is just short of two rotational periods. As with all wave packets, the dispersion can be reduced by starting with a broader packet in $\delta\theta$ but the cost would be that the initial time for a rotation large compared to $\delta\theta$ would increase.

Figure 6 is meant to show where the marked ion would be detected for different admixtures of the excited states. The first excited state corresponds to $m = N$ and gives a mark that rotates in the positive sense. The second excited state corresponds to $m = -N$ and gives a mark that rotates in a negative sense much faster than the $m = 0$ ground state. The detection time was chosen to be 1 s which is long enough for the ground state to move a distinguishable distance but short enough that the faster packets do not return to $\theta \sim 0$. At this time, the three initial states give distinct peaks. The ground state is the peak at $\eta_d/\pi \sim -0.15$, the first excited state gives the peak at ~ 0.45 , and the second excited state gives the peak at ~ -0.8 . Since the ring rotates faster for more highly excited states, contamination from them would tend to give peaks that overlap those shown. The calculation with the largest excited-state admixture has approximately 57% in the ground state. This calculation shows that somewhat large admixtures of excited states could still lead to a detectable signal corresponding to rotation of the ground state.

B. Not weak marking

This section describes the wave packet that results when the probability for more than one marked atom is not negligible.

We will define the $t = 0$ to be the time when the marking takes place. We will assume that the average number of marked ions is approximately 1.

At time $t = 0$, the part of the wave function in Eq. (14) corresponding to zero marked ions equals

$$\Psi_1^{(0)}(\Theta, 0) = \prod_{n=1}^N \cos(\alpha(\theta_n)) \Phi_0(\Theta, 0), \quad (24)$$

where the Φ_0 is defined in Eq. (18). For a general $\alpha(\theta)$ we would need to evaluate the overlap of this expression with eigenstates. However, since the ions are closely spaced compared to $\delta\theta$ of the marking, the product of cosines has almost no variation with respect to Θ . When we chose the marking width to be comparable to the ion spacing, $\delta\theta = 2\pi/N$, the variation was approximately 0.4%; the variation was approximately 10^{-6} for $\delta\theta$ twice this value. For the $\delta\theta$ used in our calculations, the product of cosines had a variation with respect to Θ that was smaller than machine precision. Thus, to an excellent approximation,

$$\Psi_1^{(0)}(\Theta, t) = A \Phi_0(\Theta, t), \quad (25)$$

where

$$A = \prod_{n=1}^N \cos(\alpha(\bar{\theta}_n)) \quad (26)$$

is a constant. Thus the zero mark part of the wave function remains in the initial eigenstate but with a probability that is decreased to A^2 .

At time $t = 0$, the part of the wave function corresponding to the n th ion being marked is

$$\Psi_1^{(1,n)}(\Theta, 0) = i A \tan(\alpha(\theta_n)) \Phi_0(\Theta, 0), \quad (27)$$

where we have used the definition from Eq. (26) and the trigonometric relation $\tan(\alpha) \cos(\alpha) = \sin(\alpha)$. Using similar operations as the previous section, the part of the wave function at later times corresponding to the n th ion being marked is

$$\Phi_n^{(1)}(\Theta, t) = i A \Phi_0(\Theta, t) \sum_{m'} e^{i\delta m' \Theta - (E_{m'} - E_m)t} \tau_{m', m}^{(n)}, \quad (28)$$

where the definitions of Eq. (19) hold and

$$\begin{aligned} \tau_{m', m}^{(n)} &= \langle m' | \tan(\alpha(\theta_n)) | m \rangle \\ &= \frac{\tau_0 \delta\theta}{\sqrt{8\pi \ln 2}} e^{i\delta m' (\bar{\theta}_n - \eta)} e^{-\delta m'^2 \delta\theta^2 / (8 \ln 2)}, \end{aligned} \quad (29)$$

where $\bar{\theta}_n = n2\pi/N$ and the last step assumed $\tan(\alpha) = \tau_0 \exp(-2 \ln(2)[\theta - \eta]^2 / \delta\theta^2)$ in the neighborhood of η . The form for $\tan(\alpha)$ was chosen for convenience of the calculation and does not have a qualitative effect on the results.

The simple form for $\tan(\alpha)$ allows for an exact evaluation of the matrix elements for more than one ion marked. There are four main changes from the expression for one marked ion: (1) the amplitude is proportional to τ_0^p where p is the number of excited atoms, (2) the spread in m modifies the term in Eq. (29) to $\exp(-\delta m'^2 \delta\theta^2 / [8p \ln 2])$, (3) the phase term is changed where the $\bar{\theta}_n$ is replaced by the average $\bar{\theta}_n$ of

the marked atoms, and (4) an extra factor of

$$S = \exp\left(-\frac{2 \ln 2}{p} \sum_{j < k}^p [\bar{\theta}_{n_j} - \bar{\theta}_{n_k}]^2\right), \quad (30)$$

which insures all of the marked ions are close to each other. Because of change (2), the wave function will disperse more quickly as more ions are marked. This would affect the measurements if the ring were allowed to rotate for more than a part of a cycle.

As with the previous section, the probe step of the process can be modeled once one has the wave function. For an average of one marked ion, the results are similar to those shown in Figs. 5 and 6.

V. CONCLUSIONS

We have performed time-dependent calculations for a ring of identical, bosonic ions in a magnetic field. These calculations can serve as a guide of what to expect for certain experimental manipulations. We showed that it is possible to pin the ring using a potential that is not rotationally symmetric.

By slowly turning off the pinning potential, the ring can be adiabatically cooled to the ground rotational state without the need for temperatures at the nK scale. We also showed that it is possible to mark an ion and measure the rotation of the ring by subsequently performing a detection of the marked ion. The time-dependent quantum calculations show that the wave function for the marked ion disperses. Thus there is an experimental trade-off in that a ring that is marked over a small $\delta\theta$ spreads quickly and vice versa.

The time and energy scales in our calculations show that the experiments will be very challenging. However, there does not appear to be a fundamental problem with detecting the rotation of a marked ring made from identical bosonic ions.

ACKNOWLEDGMENTS

We thank Tongcang Li for interesting discussions that motivated us to study this system. We also thank Zhexuan Gong and Zhangqi Yin for comments on an earlier draft of this paper. This material is based upon work supported by the National Science Foundation under Grant No. 1404419-PHY.

-
- [1] F. Wilczek, *Phys. Rev. Lett.* **109**, 160401 (2012).
 - [2] A. Shapere and F. Wilczek, *Phys. Rev. Lett.* **109**, 160402 (2012).
 - [3] T. Li, Z.-X. Gong, Z.-Q. Yin, H. T. Quan, X. Yin, P. Zhang, L.-M. Duan, and X. Zhang, *Phys. Rev. Lett.* **109**, 163001 (2012).
 - [4] P. Bruno, *Phys. Rev. Lett.* **110**, 118901 (2013).
 - [5] F. Wilczek, *Phys. Rev. Lett.* **110**, 118902 (2013).
 - [6] P. Bruno, *Phys. Rev. Lett.* **111**, 070402 (2013).
 - [7] P. Bruno, *Phys. Rev. Lett.* **111**, 029301 (2013).
 - [8] T. Li, Z.-X. Gong, Z.-Q. Yin, H. T. Quan, X. Yin, P. Zhang, L.-M. Duan, and X. Zhang, [arXiv:1212.6959](https://arxiv.org/abs/1212.6959).
 - [9] P.-J. Wang, T. Li, C. Noel, X. Zhang, and H. Häffner, [arXiv:1412.3551](https://arxiv.org/abs/1412.3551).
 - [10] B. Tabakov, F. Benito, M. Blain, C. R. Clark, S. Clark, R. A. Haltli, P. Maunz, J. D. Sterk, C. Tigges, and D. Stick, [arXiv:1501.06554](https://arxiv.org/abs/1501.06554).
 - [11] W. Paul, *Rev. Mod. Phys.* **62**, 531 (1990).
 - [12] D. Leibfried, R. Blatt, C. Monroe, and D. Wineland, *Rev. Mod. Phys.* **75**, 281 (2003).
 - [13] D. F. V. James, *Appl. Phys. B* **66**, 181 (1998).
 - [14] M. J. Madsen, W. K. Hensinger, D. Stick, J. A. Rabchuk, and C. Monroe, *Appl. Phys. B* **78**, 639 (2004).
 - [15] J. Chiaverini, R. B. Blakestad, J. Britton, J. D. Jost, C. Langer, D. Leibfried, R. Ozeri, and D. J. Wineland, *Quantum Inf. Comput.* **5**, 419 (2005).
 - [16] J. H. Wesenberg, *Phys. Rev. A* **78**, 063410 (2008).
 - [17] S. Willitsch, *Int. Rev. Phys. Chem* **31**, 175 (2012).
 - [18] G. Burmeister and K. Maschke, *Phys. Rev. B* **65**, 155333 (2002).
 - [19] T. Li (private communication).
 - [20] G. Poulsen, Y. Miroshnychenko, and M. Drewsen, *Phys. Rev. A* **86**, 051402 (2012).
 - [21] R. Rugango, J. E. Goeders, T. H. Dixon, J. M. Gray, N. B. Khanyile, G. Shu, R. J. Clark, and K. R. Brown, *New J. Phys.* **17**, 035009 (2015).
 - [22] Nonadiabatic methods might decrease the time required to switch off the pinning potential. Z. Yin (private communication). We did not explore this possibility since the time scale for a rotation experiment is longer than the time needed for adiabatic cooling.