# Chapter 7: The Interaction of Atoms with Radiation

This chapter will go into some of the tools needed for a quantitative description of how light affects atoms. We will restrict the cases where light is weak: $|E_{light}| << |E_{atom}|$

This does not necessarily mean the effects on the atom are small.

$$E_{atom} \sim \frac{e}{4\pi\varepsilon_0 a_0^2} \sim 5 \times 10^{11}\,V/m = 5 \times 10^{9}\,V/cm$$

How to estimate the E-field from light?

Energy density $\left(\frac{J}{m^3}\right) = \frac{1}{2}\varepsilon_0 E_0^2$ for $E = E_0 \cos(\omega t + \delta)$

$$I\left(\frac{W}{m^2}\right) = \frac{1}{2}\varepsilon_0 c E_0^2$$

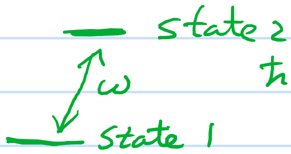
1mW focussed to $1\,mm^2 \Rightarrow I = 10^3 \frac{W}{m^2} = 10^{-1}\frac{W}{cm^2} \Rightarrow E_0 = \sqrt{\frac{2I}{\varepsilon_0 c}} \sim 10^3\,V/m$

Current high power lasers achieve $I >> 10^{16}\,W/cm^2$ by tight focus and short duration.

Because the light is weak, we will start by treating the atoms as if they only have 2 states.
Also approximate the light as classical, oscillating E-field.

For light polarized in the x-direction: $H = H_{atom} - \sum_i q_i x_i E(t)$

The Schrodinger equation: $i\hbar \frac{\partial \Psi(\vec{r}_1, \ldots, t)}{\partial t} = H\,\Psi(\vec{r}_1, \ldots, t)$




$\hbar\omega \simeq E_2 - E_1$ $\qquad$ $H_{atom}\psi_j = E_j \psi_j$

The $\psi_j$ are eigenstates of the atomic Hamiltonian and the $E_j$ are eigenenergies.

Approximation $\Psi(t) = a_1(t)\psi_1 + a_2(t)\psi_2$

How to get the equations for the $a_j(t)$?

Multiply by $\psi_1^*$ and integrate over all spatial/spin coordinates

$$\int \psi_1^* i\hbar \frac{\partial \Psi}{\partial t}\, dV = i\hbar \dot{a}_1 = \int \psi_1^* H_{atom} \Psi\, dV + \int \psi_1^* H_{las} \Psi\, dV$$
$$= E_1 a_1 + \hbar \Omega \frac{E(t)}{E_0} a_2$$

Multiply by $\psi_2^*$ and integrate gives similar equation

$$i\hbar \dot{a}_1 = E_1 a_1 + \hbar\Omega \frac{E(t)}{E_0} a_2 \quad \text{and} \quad i\hbar \dot{a}_2 = E_2 a_2 + \hbar \Omega^* \frac{E(t)}{E_0} a_1$$

with

$$\Omega = -\left[\int \psi_1^* \left(\sum_i q_i x_i\right) \psi_2\, dV\right] \frac{E_0}{\hbar} \quad \text{with} \quad E_0 = \max(E(t))$$

These equations let you obtain the $a_i(t)$ from which you can obtain all physically interesting quantities. For example?

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Math excursion of 2-state systems: Use the formalism of spin ½ systems

$$\underline{\underline{1}} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad \underline{\underline{\sigma_x}} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \underline{\underline{\sigma_y}} = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad \underline{\underline{\sigma_z}} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

Any 2x2 matrix can be written in terms of these 4

$$\underline{\underline{M}} = \begin{pmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{pmatrix} = M_0 \underline{\underline{1}} + M_x \underline{\underline{\sigma_x}} + M_y \underline{\underline{\sigma_y}} + M_z \underline{\underline{\sigma_z}} = \begin{pmatrix} M_0 + M_z & M_x - iM_y \\ M_x + iM_y & M_0 - M_z \end{pmatrix}$$

The coefficients can be obtained from the original elements

$$M_0 = \frac{M_{11}+M_{22}}{2} \quad M_x = \frac{M_{12}+M_{21}}{2} \quad M_y = \frac{M_{21}-M_{12}}{2i} \quad M_z = \frac{M_{11}-M_{22}}{2}$$

If $\underline{\underline{M}}$ is Hermitian, then $M_0, \vec{M}$ are real.

The eigenvalues are $M_\pm = M_0 \pm |\vec{M}| = M_0 \pm \sqrt{M_x^2 + M_y^2 + M_z^2}$

The coefficients can be written in vector form $|\Psi\rangle = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix}$

It is very important to know how to interpret $|\Psi\rangle$.

The most general state can be written as $|\Psi\rangle = e^{i\gamma}\begin{pmatrix}\cos(\frac{\alpha}{2})\\ e^{i\beta}\sin(\frac{\alpha}{2})\end{pmatrix}$

$$\langle\Psi|\sigma_z|\Psi\rangle = e^{-i\gamma}\begin{pmatrix}\cos(\frac{\alpha}{2}) & e^{-i\beta}\sin(\frac{\alpha}{2})\end{pmatrix}\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix} e^{i\gamma}\begin{pmatrix}\cos(\alpha/2)\\ e^{i\beta}\sin(\frac{\alpha}{2})\end{pmatrix}$$

$$= \begin{pmatrix}\cos(\frac{\alpha}{2}) & e^{-i\beta}\sin(\frac{\alpha}{2})\end{pmatrix}\begin{pmatrix}\cos(\alpha/2)\\ -e^{i\beta}\sin(\alpha/2)\end{pmatrix} = \cos^2(\tfrac{\alpha}{2}) - \sin^2(\tfrac{\alpha}{2})$$

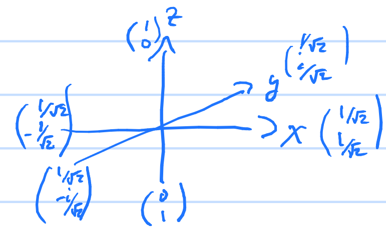
Students should do the other cases

$\langle\Psi|\underline{\underline{1}}|\Psi\rangle = 1$

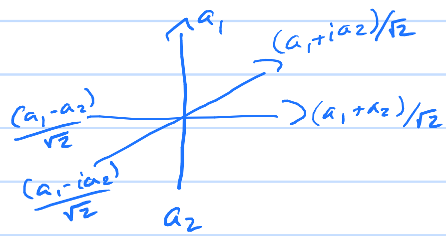
$\langle\Psi|\underline{\underline{\sigma}}_x|\Psi\rangle = \sin(\alpha)\cos(\beta)$

$\langle\Psi|\underline{\underline{\sigma}}_y|\Psi\rangle = \sin(\alpha)\sin(\beta)$

$\langle\Psi|\underline{\underline{\sigma}}_z|\Psi\rangle = \cos(\alpha)$




Block Sphere (See Fig 7.2)




End of math digression

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We will be mainly interested in cases where $E(t) = E_0\, F(t)\cos(\omega t)$
$F(t)$ is a slowly evolving function of time

Our weak laser approximation means $\Omega \ll \omega$

The two equations from above can be written as

$$i\frac{d}{dt}\begin{pmatrix}a_1\\ a_2\end{pmatrix} = \left(M_0\,\underline{\underline{1}} + \vec{M}(t)\cdot\vec{\sigma}\right)\begin{pmatrix}a_1\\ a_2\end{pmatrix}$$

$$M_0 = \frac{E_2+E_1}{2\hbar} \qquad M_x = \frac{\Omega+\Omega^*}{2}F(t)\cos(\omega t) \qquad M_y = \frac{\Omega^*-\Omega}{2i}F(t)\cos(\omega t) \qquad M_z = \frac{E_1-E_2}{2\hbar}$$

The part with $M_0$ has no meaning because we can define the 0 of energy so $E_2 + E_1 = 0$

In general, the solution of this differential equation can not be found analytically. So will look at special cases to get an idea of how it works.

Perturbative case: $a_1(-\infty) = 1 \quad a_2(-\infty)=0$ and $|a_2(t)| \ll 1$

Define $\omega_0 = (E_2 - E_1)/\hbar$

$$i\dot{a}_1 = -\frac{\omega_0}{2} a_1 + \Omega F(t)\cos(\omega t) a_2 \qquad i\dot{a}_2 = \frac{\omega_0}{2} a_2 + \Omega^* F(t)\cos(\omega t) a_1$$

Account for the fast oscillation $a_1 = C_1 e^{i\omega_0 t/2}$ and $a_2 = C_2 e^{-i\omega_0 t/2}$

$$i\dot{C}_1 = \Omega F(t)\cos(\omega t) e^{-i\omega_0 t} C_2 \qquad i\dot{C}_2 = \Omega^* F(t)\cos(\omega t) e^{i\omega_0 t} C_1$$

Formally can write these as

$$C_1(t) = C_1(-\infty) - i\Omega \int^{t} F(t')\cos(\omega t') e^{-i\omega_0 t'} C_2(t')\,dt'$$
$$C_2(t) = C_2(-\infty) - i\Omega^* \int_{-\infty}^{t} F(t')\cos(\omega t') e^{i\omega_0 t} C_1(t')\,dt'$$

Perturbation theory is just the recursive substitution of solutions

$$C_1^{(n)}(t) = C_1(-\infty) - i\Omega \int_{-\infty}^{t} F(t')\cos(\omega t') e^{-i\omega_0 t'} C_2^{(n-1)}(t')\,dt'$$
$$C_2^{(n)}(t) = C_2(-\infty) - i\Omega^* \int_{-\infty}^{t} F(t')\cos(\omega t') e^{i\omega_0 t'} C_1^{(n-1)}(t')\,dt'$$

When starting $C_1(-\infty)=1 \quad C_2(-\infty)=0$, then $C_1$ only has even powers of $|\Omega|$ and $C_2$ only has $\Omega^*$ times even powers of $|\Omega|$

$$C_1^{(0)} = 1 \qquad C_2^{(0)} = 0 \Rightarrow C_1^{(1)} = 1 \quad \text{and} \quad C_2^{(1)}(t) = -i\Omega^* \int_{-\infty}^{t} F(t')\cos(\omega t') e^{i\omega_0 t'}\,dt'$$

Next order

$$C_1^{(2)}(t) = 1 - |\Omega|^2 \int_{-\infty}^{t} F(t')\cos(\omega t') e^{-i\omega_0 t'} \left[\int_{-\infty}^{t'} F(t'')\cos(\omega t'') e^{i\omega_0 t''}\,dt''\right] dt'$$
$$C_2^{(2)}(t) = C_2^{(1)}(t)$$

Now look in detail at the 1st order term

$$C_2^{(1)}(t) = -i\frac{\Omega^*}{2}\int_{-\infty}^{t} F(t') e^{i(\omega+\omega_0)t'}\,dt' + -i\frac{\Omega^*}{2}\int_{-\infty}^{t} F(t') e^{i(\omega_0-\omega)t'}\,dt'$$

Because $F(t)$ is slowly varying and $\omega$ is so large, the first term is effectively 0.

$\omega, \omega_0$ of order $10^{15}\,s^{-1}$, but $F(t)$ variation ~ ns to us

Dropping the first term is the <u>rotating wave approximation</u>

An example case: $F(t) = e^{-t^2/\tau^2}$

$$C_2^{(1)}(\infty) = -i\frac{\Omega^*}{2}\int_{-\infty}^{\infty} e^{-t'^2/\tau^2} e^{i(\omega_0-\omega)t'}dt' = -i\frac{\Omega^*}{2}\sqrt{\pi}\,\tau\, e^{-(\omega_0-\omega)^2\tau^2/4}$$

The probability for the atom to have transitioned to state 2

$$P_2 = |C_2|^2 = \frac{|\Omega|^2}{4}\pi\tau^2 e^{-(\omega_0-\omega)^2\tau^2/2} = \frac{|\langle 1|qx|2\rangle|^2}{4\hbar^2}\pi\tau^2 E_0^2 e^{-(\omega_0-\omega)^2\tau^2/2}$$

This is strongly peaked when $\omega = \omega_0 = \frac{E_2 - E_1}{\hbar}$ (No photons! Classical E+M)

For any $F(t)$, $P_2$ is proportional to the Fourier transform of the electric field.

$$E(\omega_0) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} E(t)e^{i\omega_0 t}dt \Rightarrow P_2 = 2\pi\frac{|\langle 1|qx|2\rangle|^2}{\hbar^2}|E(\omega_0)|^2$$

Now turn to the case where $F(t) = 1$ (monochromatic light)

$$i\dot{C}_1 = \Omega\cos(\omega t)e^{-i\omega_0 t}C_2 \qquad i\dot{C}_2 = \Omega^*\cos(\omega t)e^{i\omega_0 t}C_1$$

Make the rotating wave approximation

$$i\dot{C}_1 = \frac{\Omega}{2}e^{i(\omega-\omega_0)t}C_2 \qquad i\dot{C}_2 = \frac{\Omega^*}{2}e^{-i(\omega-\omega_0)t}C_1$$

Define $\delta = \omega - \omega_0$ and use the variables $C_1 = \tilde{C}_1 e^{i\delta t/2}$ $C_2 = \tilde{C}_2 e^{-i\delta t/2}$

$$i\dot{\tilde{C}}_1 = \frac{\delta}{2}\tilde{C}_1 + \frac{\Omega}{2}\tilde{C}_2 \qquad i\dot{\tilde{C}}_2 = -\frac{\delta}{2}\tilde{C}_2 + \frac{\Omega^*}{2}\tilde{C}_1$$

For bound states we can always define the phases of the states so $\Omega = \Omega^*$

$$i\frac{d}{dt}\begin{pmatrix}\tilde{C}_1\\ \tilde{C}_2\end{pmatrix} = \left[\frac{\delta}{2}\sigma_z + \frac{\Omega}{2}\sigma_x\right]\begin{pmatrix}\tilde{C}_1\\ \tilde{C}_2\end{pmatrix}$$

Take derivative of equations with respect to $t$ again

$$\ddot{\tilde{C}}_1 = -\left(\frac{\delta^2}{4} + \frac{|\Omega|^2}{4}\right)\tilde{C}_1 \quad \text{and} \quad \ddot{\tilde{C}}_2 = -\left(\frac{\delta^2}{4} + \frac{|\Omega|^2}{4}\right)\tilde{C}_2$$

Need to account for the boundary conditions

$\tilde{C}_1(0) = 1 \qquad \dot{\tilde{C}}_1(0) = -i\delta/2 \qquad \tilde{C}_2(0) = 0 \qquad \dot{\tilde{C}}_2(0) = -i\frac{\Omega^*}{2}$

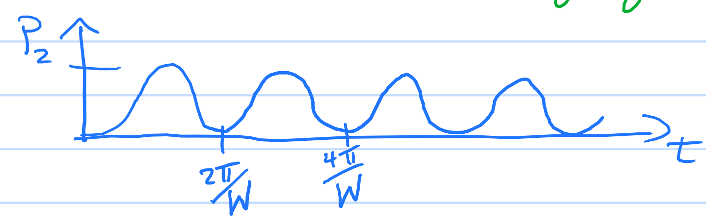
$W^2 \equiv \delta^2 + \Omega^2$

$$\tilde{C}_1(t) = \cos\left(\frac{W}{2}t\right) - \frac{i\delta}{W}\sin\left(\frac{W}{2}t\right) \qquad \tilde{C}_2 = -\frac{i\Omega^*}{W}\sin\left(\frac{Wt}{2}\right)$$

The probability to be in state 2 $\quad P_2 = |\tilde{C}_2|^2 = \frac{|\Omega|^2}{W^2}\sin^2\left(\frac{Wt}{2}\right)$

When exactly on resonance $W = |\Omega|$ The time for the population to go back 100% to state 1 is $T = 2\pi/\Omega$

$\Omega$ called the Rabi frequency




$\max(P_2) = \frac{|\Omega|^2}{\delta^2 + |\Omega|^2}$ has Lorentzian shape

The more general solution is

$$\tilde{C}_1(t) = \tilde{C}_1(0)\left[\cos\left(\frac{W}{2}t\right) - \frac{i\delta}{W}\sin\left(\frac{W}{2}t\right)\right] - \frac{i\Omega}{W}\tilde{C}_2(0)\sin\left(\frac{W}{2}t\right)$$
$$\tilde{C}_2(t) = \tilde{C}_2(0)\left[\cos\left(\frac{W}{2}t\right) + \frac{i\delta}{W}\sin\left(\frac{W}{2}t\right)\right] - \frac{i\Omega^*}{W}\tilde{C}_1(0)\sin\left(\frac{W}{2}t\right)$$

Example: How does $\langle\vec{\sigma}\rangle$ behave when starting $\tilde{C}_1(0) = 1$?

a) Case 1, no detuning $\Rightarrow \delta = 0$

$\tilde{C}_1(t) = \cos\left(\frac{\Omega}{2}t\right) \qquad \tilde{C}_2 = -i\frac{\Omega^*}{|\Omega|}\sin\left(\frac{\Omega}{2}t\right)$

$\alpha = \frac{\Omega}{2}t \qquad e^{i\beta} = -i\frac{\Omega^*}{|\Omega|} = e^{-i(\pi/2 + \zeta)} \qquad$ where $\Omega = |\Omega|e^{i\zeta}$

$\pi$ pulse has $\frac{\Omega}{2}T = \frac{\pi}{2} \qquad \tilde{C}_1(T) = 0 \qquad \tilde{C}_2(T) = -i \qquad$ when $\zeta = 0$

The general case has $\quad \tilde{C}_1(T) = -i\tilde{C}_2(0)$ and $\tilde{C}_2(T) = -i\tilde{C}_1(0)$

The $\pi$ pulse swaps the 1 and 2 states but with an overall $-i$ factor

$\pi/2$ pulse has $\frac{\Omega}{2}T = \frac{\pi}{4} \qquad \tilde{C}_1(T) = 1/\sqrt{2} \qquad \tilde{C}_2(T) = -i/\sqrt{2} \qquad$ when $\zeta = 0$

The general case has $\quad \tilde{C}_1(T) = \frac{\tilde{C}_1(0)}{\sqrt{2}} - i\frac{\tilde{C}_2(0)}{\sqrt{2}} \qquad \tilde{C}_2(T) = -i\frac{\tilde{C}_1(0)}{\sqrt{2}} + \frac{\tilde{C}_2(0)}{\sqrt{2}}$

This case splits the 1 into equal amounts of 1 and 2 and similarly for 2. This can lead to max interference

When a laser excites a transition from $\psi_1$ to $\psi_2$, the width of the transition (spread in frequencies) is partly due to the frequency spread of the laser. Fourier transform limited pulse has $\Delta f$ inversely proportional to $\Delta t$.

Example from above: $E(t) = E_0\, F(t) \cos(\omega t)$ with $F(t) = e^{-t^2/\tau^2}$

$$C_2^{(1)}(\infty) = -i\frac{\Omega^*}{2}\sqrt{\pi}\,\tau\, e^{-(\omega_0-\omega)^2\tau^2/4} \qquad P_2 = |C_2|^2 = \frac{|\Omega|^2}{4}\pi\tau\, e^{-(\omega_0-\omega)^2\tau^2/2}$$

As $\tau$ increases, the spread in $\omega$ decreases.

Ramsey figured out way to gain accuracy without having to keep laser on the whole time (he actually did the experiment with microwaves). Have two pulses separated by a time $T$.

$$F(t) = F_0(t) + F_0(t-T)$$

$$C_2^{(1)}(\infty) = -i\frac{\Omega^*}{2}\left[\int_{-\infty}^{\infty} F_0(t')\, e^{i(\omega_0-\omega)t'} dt' + \int_{-\infty}^{\infty} F_0(t'-T)\, e^{i(\omega_0-\omega)t'} dt'\right]$$

$$= -i\frac{\Omega^*}{2}\left[1 + e^{i(\omega_0-\omega)T}\right]\int_{-\infty}^{\infty} F_0(t')\, e^{i(\omega_0-\omega)t'} dt'$$

$$P_{2,2\,\text{pulses}} = P_{2,1\,\text{pulse}}\; 4\cos^2\left[\frac{(\omega_0-\omega)T}{2}\right]$$

The extra, fast oscillating term called Ramsey fringes. The $P_{2,2\,\text{pulses}}$ goes from a max when $\omega_0 = \omega$ to $0$ when $\frac{\omega-\omega_0}{2}T = \pi/2$. This implies one can measure frequency differences of $\sim 1/T$. Current use includes optical fountains.

This technique is also important in cases where the energy separation of the states is drifting.

In 1927, John von Neumann and (independently) Lev Landau introduced the idea of the density matrix. This is useful when the quantum system is not in a pure state or a state is (for example) emitting photons.

For a pure state, the density matrix is defined as

$\Psi = a_1 \psi_1 + a_2 \psi_2$

$\hat{\rho} \equiv |\Psi\rangle\langle\Psi|$ in matrix form $\underline{\underline{\rho}} = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} \begin{pmatrix} a_1^* & a_2^* \end{pmatrix} = \begin{pmatrix} |a_1|^2 & a_1 a_2^* \\ a_1^* a_2 & |a_2|^2 \end{pmatrix}$

The density matrix satisfies $\frac{\partial \hat{\rho}}{\partial t} = \frac{\partial |\Psi\rangle}{\partial t}\langle\Psi| + |\Psi\rangle \frac{\partial \langle\Psi|}{\partial t}$

$= -\frac{i}{\hbar}(\hat{H}\hat{\rho} - \hat{\rho}\hat{H})$

Matrix form $\frac{d\underline{\underline{\rho}}}{dt} = -\frac{i}{\hbar}(\underline{\underline{H}}\,\underline{\underline{\rho}} - \underline{\underline{\rho}}\,\underline{\underline{H}})$

Because $\rho$ is Hermitian $\underline{\underline{\rho}} = \frac{1}{2}(\underline{\underline{1}} + \vec{\rho}\cdot\vec{\sigma})$ with $\vec{\rho}$ all real

$\rho_x = (a_1 a_2^* + a_1^* a_2)$ $= \langle \sigma_x \rangle$

$\rho_y = -i\,(a_1^* a_2 - a_1 a_2^*)$ $= \langle \sigma_y \rangle$

$\rho_z = (|a_1|^2 - |a_2|^2)$ $= \langle \sigma_z \rangle$

One of the interesting aspects of the density matrix is the expectation values of operators turn into traces with $\rho$.

$\rho_{ij} = a_i a_j^*$

$\langle \hat{Q} \rangle = \sum_{ij} a_i^* Q_{ij} a_j = \sum_{ij} Q_{ij}\rho_{ji} = \sum_i \left(\sum_j Q_{ij}\rho_{ji}\right) = Tr(Q\rho) = Tr(\rho Q)$

People often perform transformations on $\rho$ to take out oscillation from H atom (Foot does)

$\psi = c_1(t)e^{-iE_1 t/\hbar}\psi_1 + c_2(t)e^{-iE_2 t/\hbar}\psi_2$ $\rho_{ij} = c_i c_j^*$

Since $a_j(t) = c_j(t)e^{-iE_j t/\hbar}$

$\rho_{ij}^{(0)} = a_i a_j^* = c_i c_j^* e^{-i(E_i - E_j)t/\hbar} = \rho_{ij} e^{-i(E_i - E_j)t/\hbar}$

This will give a modified set of equations because we've incorporated part of the Hamiltonian into $\rho$.

The nicest aspect of $\rho$ is that it gives us a simple method for incorporating photon emission. Atoms can emit millions of photons per second!

An important aspect of the physics is missing: the atom can spontaneously emit a photon if it is in the excited state $\Psi_2$. There should be some sort of competition between the half life (proportional to $1/\Gamma$) and the time needed for the atom to get to $P_{2,max}$ ($\pi/W$)

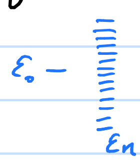
Physics excursion into why states decay.

What is the same: $Na(3p) \rightarrow Na(3s) + hf$ (photon emission), $n \rightarrow p + e^+ + \nu_e$ (beta decay), ${}^{238}_{92}U \rightarrow {}^{234}_{90}Th + \alpha$ (alpha emission), $He(2s^2) \rightarrow He^+(1s) + \varepsilon s$ (electron emission), etc ....

A single state is at the same energy as a continuum of states. All of these states have an exponential decay of population with a characteristic lifetime that depends on the system. Lorentzian energy dependence of decay products

A simple model: One state $\Psi_0$, a discretized continua of states $\Psi_n$ ($n = 1, \ldots 2N-1$, The state $\Psi_0$ can interact with the $\Psi_n$ and vice versa, The energy of $\Psi_0$ is $\varepsilon_0$ and the energy of $\Psi_n$ is $\varepsilon_n = \varepsilon_0 + \Delta\varepsilon \cdot (n - N)$

The coupling between $\Psi_0$ and $\Psi_n$ are all same




$$i\hbar \dot{a}_0 = \varepsilon_0 a_0 + \lambda \sqrt{\Delta\varepsilon} \sum_{n=1}^{2N-1} a_n \qquad i\hbar \dot{a}_n = \varepsilon_n a_n + \lambda^* \sqrt{\Delta\varepsilon}\, a_0$$

Make the usual transformation to pull out the fast dependence

$$a_j = e^{-i\varepsilon_j t/\hbar} C_j$$

$$i\dot{C}_0 = \lambda \sqrt{\frac{\Delta\omega}{\hbar}} \sum_{n=1}^{2N-1} C_n e^{-i(\omega_n - \omega_0)t} \qquad i\dot{C}_n = \lambda^* \sqrt{\frac{\Delta\omega}{\hbar}}\, C_0 e^{i(\omega_n - \omega_0)t}$$

Defined $\omega_j \equiv \varepsilon_j/\hbar$

Use the initial conditions $C_0(0)=1$ and $C_n(0)=0$

$$C_n(t) = -i\lambda^* \sqrt{\frac{\Delta\omega}{\hbar}} \int_0^t C_0(t')\, e^{i(\omega_n-\omega_0)t'} dt'$$

$$\dot{C}_0(t) = -|\lambda|^2 \frac{\Delta\omega}{\hbar} \sum_{n=1}^{2N-1} \int_0^t e^{i(\omega_n-\omega_0)(t'-t)} C_0(t')\, dt'$$

If we could solve the 2nd equation, we could plug that result into the first equation to get the $C_n(t)$ as well

For a true continuum, $N \to \infty$ and then $\Delta\omega \to 0$

$$\dot{C}_0(t) = -|\lambda|^2 \frac{1}{\hbar} \int_{-\infty}^{\infty} \int_0^t e^{i\omega(t'-t)} C_0(t')\, dt'\, d\omega$$

$$= -|\lambda|^2 \frac{1}{\hbar} \int_0^t 2\pi\, \delta(t'-t)\, C_0(t')\, dt'$$

$$\dot{C}_0(t) = -\frac{\pi|\lambda|^2}{\hbar} C_0(t) \qquad C_0(t) = e^{-\Gamma t/2} \qquad \Gamma = \frac{2\pi\lambda^2}{\hbar}$$

$$C_n(\infty) = -i\lambda^* \sqrt{\frac{\Delta\omega}{\hbar}} \int_0^\infty e^{-\frac{\Gamma t'}{2} + i(\omega_n-\omega_0)t'} dt' = -i\lambda^* \sqrt{\frac{\Delta\omega}{\hbar}} \frac{1}{\frac{\Gamma}{2} - i(\omega_n-\omega_0)}$$

$$|C_n(\infty)|^2 = \Delta\omega \frac{\Gamma/2\pi}{(\omega_n-\omega_0)^2 + (\Gamma/2)^2}$$

Recap: the model gives a population in state 0 that decays exponentially

$$P_0(t) = |C_0(t)|^2 = e^{-\Gamma t}$$

The continuum has a Lorentzian population once the state has fully decayed

$$P_n(\infty) = \Delta\omega\, \frac{\Gamma}{2\pi} \Big/ \left[(\omega-\omega_0)^2 + (\Gamma/2)^2\right]$$

All of the population is in the continuum $\sum_n P_n = \int d\omega \frac{\Gamma/2\pi}{(\omega-\omega_0)^2+(\Gamma/2)^2} = 1$

Fermi's Golden Rule: $\Gamma = \frac{2\pi}{\hbar} |\langle f|H|i\rangle|^2 \rho$ ← density of states

Model: $\langle f|H|i\rangle = \lambda\sqrt{\Delta\varepsilon}$, $\rho = 1/\Delta\varepsilon$, $\Gamma = \frac{2\pi}{\hbar} \frac{|\lambda|^2 \cancel{\Delta\varepsilon}}{\cancel{\Delta\varepsilon}}$ ✓

End of physics excursion

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The optical Bloch equations are developed from the equations for the $\tilde{c}$. Start with the undamped equations. From above

$$i\,\dot{\tilde{c}}_1 = \frac{\delta}{2}\tilde{c}_1 + \frac{\Omega}{2}\tilde{c}_2 \qquad \text{and} \qquad i\,\dot{\tilde{c}}_2 = -\frac{\delta}{2}\tilde{c}_2 + \frac{\Omega}{2}\tilde{c}_1$$

Assumed $\Omega^* = \Omega$

More compact form $\quad i\frac{d}{dt}\begin{pmatrix}\tilde{c}_1\\ \tilde{c}_2\end{pmatrix} = \left[\frac{\delta}{2}\underline{\underline{\sigma_z}} + \frac{\Omega}{2}\underline{\underline{\sigma_x}}\right]\begin{pmatrix}\tilde{c}_1\\ \tilde{c}_2\end{pmatrix}$

Using $\tilde{\rho}_{ij} = \tilde{c}_i\,\tilde{c}_j^*$ $\qquad \frac{d\tilde{\rho}}{dt} = -i\left[\frac{\delta}{2}\underline{\underline{\sigma_z}} + \frac{\Omega}{2}\underline{\underline{\sigma_x}}, \tilde{\rho}\right]$

Finally define $\tilde{\rho} = \frac{1}{2}\left(\underline{\underline{1}} + \tilde{\rho}_x\underline{\underline{\sigma_x}} + \tilde{\rho}_y\underline{\underline{\sigma_y}} + \tilde{\rho}_z\underline{\underline{\sigma_z}}\right)$

Detailed math steps that can be skipped

$[\sigma_x, \sigma_y] = 2i\,\sigma_z$ etc

$$-i\left[\frac{\delta}{2}\sigma_z + \frac{\Omega}{2}\sigma_x, \tilde{\rho}\right] = -i\frac{\delta}{2}[\sigma_z, \tilde{\rho}] - i\frac{\Omega}{2}[\sigma_x, \tilde{\rho}]$$
$$= -i\frac{\delta}{2}\left(-i\sigma_x\tilde{\rho}_y + i\sigma_y\tilde{\rho}_x\right) - i\frac{\Omega}{2}\left(i\sigma_z\tilde{\rho}_y - i\sigma_y\tilde{\rho}_z\right)$$

This gives the differential equations for the $\tilde{\rho}_i$

$$\frac{d\tilde{\rho}_x}{dt} = -\delta\,\tilde{\rho}_y \qquad \frac{d\tilde{\rho}_y}{dt} = \delta\,\tilde{\rho}_x - \Omega\,\tilde{\rho}_z \qquad \frac{d\tilde{\rho}_z}{dt} = \Omega\,\tilde{\rho}_y$$

Connect to the book notation $\quad u = \tilde{\rho}_x \quad v = -\tilde{\rho}_y \quad w = \tilde{\rho}_z$

This set of equations has the form $\frac{d\vec{\tilde{\rho}}}{dt} = \vec{W}\times\vec{\tilde{\rho}}$ with $\vec{W} = (\Omega, 0, \delta)$ Go look at the differential equation for $\tilde{c}$.

Implications
1) $\vec{\tilde{\rho}}\cdot\vec{\tilde{\rho}} = \text{constant}$
2) For a density matrix from a pure state, $\vec{\tilde{\rho}}\cdot\vec{\tilde{\rho}} = 1$
3) $\vec{\tilde{\rho}}\cdot\hat{W} = \text{constant}$
4) The part of $\vec{\tilde{\rho}}$ perpendicular to $\vec{W}$ rotates with angular frequency $|\vec{W}| = \sqrt{\delta^2+\Omega^2}$. Where did we see that before?
5) Interpretation $\langle\sigma_x\rangle = \tilde{\rho}_x$, etc

To this point, the density matrix is cute but nothing new. However, it will let us plausibly extend to the case including photon emission.


In a two state system, a photon emission has the atom go from 2 to 1 with a rate $\Gamma$

From spontaneous emission $\frac{d\tilde{\rho}_{22}}{dt} = -\Gamma \tilde{\rho}_{22}$ and $\frac{d\tilde{\rho}_{11}}{dt} = \Gamma \tilde{\rho}_{22}$

Combine to get

$$\frac{d\tilde{\rho}_z}{dt} = \Gamma \tilde{\rho}_{22} - -\Gamma \tilde{\rho}_{22} = 2\Gamma \tilde{\rho}_{22} = -\Gamma (\tilde{\rho}_z - 1)$$

In the $\tilde{\rho}_x$ and $\tilde{\rho}_y$, they contain one factor of $\tilde{c}_2$ or $\tilde{c}_2^*$, both of which should have an $e^{-\Gamma t/2}$

$$\frac{d\tilde{\rho}_x}{dt} = -\frac{\Gamma}{2} \tilde{\rho}_x \qquad \frac{d\tilde{\rho}_y}{dt} = -\frac{\Gamma}{2} \tilde{\rho}_y$$

Combine with the results of the laser

$$\frac{d\tilde{\rho}_x}{dt} = -\delta \tilde{\rho}_y - \frac{\Gamma}{2} \tilde{\rho}_x \qquad \frac{d\tilde{\rho}_y}{dt} = \delta \tilde{\rho}_x - \Omega \tilde{\rho}_z - \frac{\Gamma}{2} \tilde{\rho}_y \qquad \frac{d\tilde{\rho}_z}{dt} = \Omega \tilde{\rho}_y - \Gamma (\tilde{\rho}_z - 1)$$

This brings some qualitative changes to the dynamics.

1) Instead of oscillating forever, the $\vec{\rho}$ will settle into a steady state
2) The length of $\vec{\rho}$ is less than $1 > \vec{\rho} \cdot \vec{\rho}$. $\Rightarrow$ not a wave function
3) Easy to think about rates.
4) Can think of Quantum Zeno effect

Look at plots of $\vec{\rho}$ for different combinations of $\delta, \Omega, \Gamma$

The steady state solution is important because it will tell us how fast the atom will scatter photons. Get the steady state equations by setting all the $d\tilde{\rho}_i/dt = 0$

$$0 = -\delta \tilde{\rho}_y - \frac{\Gamma}{2} \tilde{\rho}_x \qquad 0 = \delta \tilde{\rho}_x - \Omega \tilde{\rho}_z - \frac{\Gamma}{2} \tilde{\rho}_y \qquad 0 = \Omega \tilde{\rho}_y - \Gamma \tilde{\rho}_z + \Gamma$$

$$\begin{pmatrix} \tilde{\rho}_x \\ \tilde{\rho}_y \\ \tilde{\rho}_z \end{pmatrix} = \frac{1}{\delta^2 + \frac{\Omega^2}{2} + (\Gamma/2)^2} \begin{pmatrix} \Omega \delta \\ -\Omega \Gamma/2 \\ \delta^2 + (\Gamma/2)^2 \end{pmatrix}$$

The probability to be in state 2 $\quad \tilde{\rho}_{22} = \frac{1}{2}(1 - \tilde{\rho}_z) = \frac{\Omega^2/4}{\delta^2 + \frac{\Omega^2}{2} + (\Gamma/2)^2}$

This is a Lorentzian in photon frequency $\delta = \omega - \omega_0$. The FWHM increases with the laser intensity

$$\Gamma_{FWHM} = 2\sqrt{\Omega^2/2 + (\Gamma/2)^2}$$

For small intensity, the population is proportional to $\Omega^2$ which is proportional to laser intensity

The rate that photons are scattered is $\Gamma_{scat} = \tilde{\rho}_{22}\Gamma$ Why?

$$\Gamma_{scat} = \frac{\Gamma\Omega^2/4}{\delta^2 + \frac{\Omega^2}{2} + (\Gamma/2)^2} \simeq \frac{\Gamma\Omega^2/4}{\delta^2 + (\Gamma/2)^2} \quad \text{when } \Omega \ll \Gamma$$

This will be an important quantity for understanding optical molasses.

The absorption cross section can be obtained now

$$(\rho_{11} - \rho_{22})\,\sigma(\omega)\,\frac{I(\omega)}{\hbar\omega} = \Gamma_{scat} \qquad \sigma(\omega) = \frac{\hbar\omega}{I(\omega)\,2\tilde{\rho}_{z}}\,\Gamma_{scat}$$

$$\sigma(\omega) \simeq \frac{\hbar\omega}{\frac{1}{2}\epsilon_0 c E_0^2}\,\frac{\Gamma\Omega^2/4}{\delta^2 + (\Gamma/2)^2} \qquad \text{with} \qquad \Omega = \frac{\langle 1|e x|2\rangle}{\hbar}E_0 \equiv e X_{12} E_0/\hbar$$

$$= \frac{\omega_0 \pi e^2 X_{12}^2}{\hbar \epsilon_0 c}\,\frac{\Gamma/2\pi}{(\omega-\omega_0)^2 + (\Gamma/2)^2}$$

Relations derived elsewhere (see book Chap 7.2, Eq 7.23 for example)

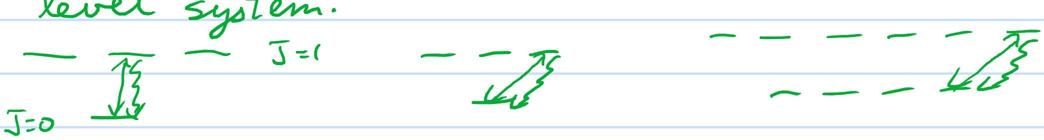
$$D_{12}^2 = |X_{12}|^2 + |Y_{12}|^2 + |Z_{12}|^2$$

$$\Gamma = A_{21} = \frac{g_1}{g_2}\frac{\omega^3}{3\pi\epsilon_0\hbar c^3} D_{12}^2$$

The $g_1$ and $g_2$ count the number of states at energy 1 and 2

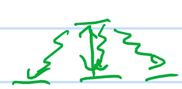
$$\sigma(\omega) = \frac{g_2}{g_1}\frac{\pi^2 c^2}{\omega_0^2}\,\Gamma\,\frac{\Gamma/2\pi}{(\omega-\omega_0)^2 + (\Gamma/2)^2}$$

There are modifications when other levels participate. The $g_2/g_1$ has the value 3 when the atom has orientation for perfect two level system.

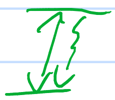



$M_1 = J_1$ and $M_2 = J_2$ with $J_2 = J_1 + 1$

Other orientations have leakage/optical pumping



When the light is perfectly on resonance, $\delta = \omega - \omega_0 = 0$, then the cross section only depends on the frequency and $g$'s



$$\sigma = \frac{g_2}{g_1}\frac{2\pi c^2}{\omega_0^2} = \frac{g_2}{g_1}\frac{\lambda_0^2}{2\pi} \qquad \lambda_0 = \frac{2\pi c}{\omega_0}$$

The power broadening is seen in the denominator

$$\delta^2 + \Omega^2/2 + (\Gamma/2)^2 = (\omega-\omega_0)^2 + (\Gamma/2)^2\left[1 + 2\Omega^2/\Gamma^2\right]$$

The combo $\dfrac{\Omega^2/2}{(\Gamma/2)^2}$ can be written as $I/I_{sat}$

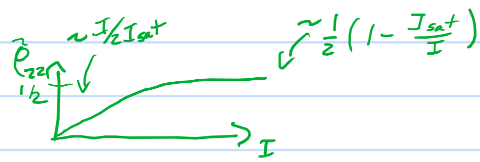
$$\sigma = \frac{\hbar\omega}{I}\,\Gamma/2\left(\frac{\Omega^2/2}{(\Gamma/2)^2}\right) \quad \Rightarrow \quad I_{sat} = \frac{\hbar\omega\Gamma}{2\sigma}$$

$$\delta^2 + \frac{\Omega^2}{2} + (\Gamma/2)^2 = (\omega-\omega_0)^2 + (\Gamma/2)^2\left[1 + I/I_{sat}\right]$$

Write the excited state population in terms of $I_{sat}$

$$\tilde{\rho}_{22} = \frac{\Omega^2/4}{\delta^2 + \Omega^2/2 + (\Gamma/2)^2} = \frac{(\Gamma/2)^2\,\frac{1}{2}\frac{I}{I_{sat}}}{(\omega-\omega_0)^2 + (\Gamma/2)^2(1 + I/I_{sat})}$$

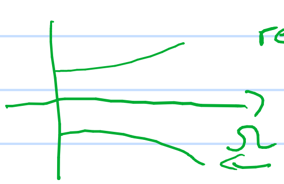
Exactly on resonance $\quad \tilde{\rho}_{22} = \dfrac{I/2}{I + I_{sat}}$

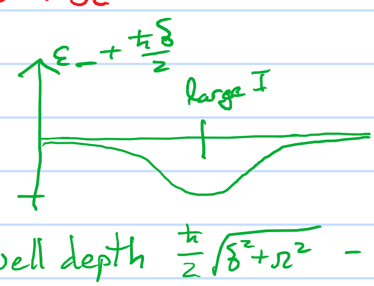



Last topic is important for trapping atoms in a dipole trap. The induced dipole interacts with the laser to attract or repel the atom from regions of high intensity. Can obtain the effect by $\vec{P}_{ind}\cdot\vec{E}$ or from quasi-energies.

$$i\hbar\frac{d}{dt}\begin{pmatrix}\tilde{c}_1\\ \tilde{c}_2\end{pmatrix} = \frac{\hbar}{2}\begin{pmatrix}\delta & \Omega\\ \Omega & -\delta\end{pmatrix}\begin{pmatrix}\tilde{c}_1\\ \tilde{c}_2\end{pmatrix}$$

The eigenvalues of the matrix are $\varepsilon_\pm = \pm\frac{\hbar}{2}\sqrt{\delta^2 + \Omega^2}$




Typically $\delta \gg \Omega$ $\qquad \varepsilon_\pm = \pm\frac{\hbar}{2}\left(\delta + \frac{\Omega^2}{2\delta}\right)$

The Quantum Zeno effect arises because the probability for going to state 2, at early times, is proportional to $t^2$. If $\delta = 0$

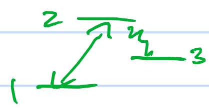
$$P_2(t) = \sin^2\left(\frac{\Omega t}{2}\right)$$

After a time $\Delta t$ do a measurement $\quad P_2 \cong \frac{\Omega^2 \Delta t^2}{4}$

After another time $\Delta t$ do a measurement $\quad P_2 \cong 2\,\frac{\Omega^2 \Delta t^2}{4}$

After $N$ measurements $\quad P_2 = \frac{1}{4}\Omega^2 \Delta t \, N \Delta t = \frac{1}{4}\Omega^2 T_f \, \frac{T_f}{N}$

final time

Compare this to the case where  state 2 decays to distinguishable state 3.

For perfectly on resonance $\quad \delta = 0$

$$\dot{\tilde{C}}_1 = -\frac{i\Omega}{2}\tilde{C}_2 \qquad \dot{\tilde{C}}_2 = -\frac{\Gamma}{2}\tilde{C}_2 - \frac{i\Omega}{2}\tilde{C}_1$$

Take a $2^{nd}$ derivative and do some manipulations to get rid of $\tilde{C}_2$

$$\ddot{\tilde{C}}_1 + \frac{\Gamma}{2}\dot{\tilde{C}}_1 + \frac{\Omega^2}{4}\tilde{C}_1 = 0$$

There are two solutions with the form $\quad \tilde{C}_1 = e^{-\alpha t}$
with

$$\alpha = \frac{\Gamma/2 \pm \sqrt{(\Gamma/2)^2 - \Omega^2}}{2}$$

For Quantum Zeno effect assume $\quad \Gamma \gg \Omega$
This gives

$\alpha \cong \Gamma/2 \quad$ corresponds to state 2 decaying

$\alpha \cong \Omega^2/\Gamma \quad$ corresponds to state 1 decaying by coupling to state 2 which is being detected with rate $\Gamma$

The probability to transition out of state 1

$$1 - P_1 = 1 - e^{-2\alpha t}$$
$$= 2\alpha t = \Omega^2 \frac{2}{\Gamma} T_f$$

This is the same form as the simple treatment above with $1/\Gamma$ in the role of $T_f/N$. As the detection rate, $\Gamma$, increases, the probability to make a transition decreases.