Chap 12: The Paul Trap

Neutral atoms can be trapped in many ways: magnetic traps, optical traps, MOT's, ...

Charged particles are both simpler and harder because of the large forces that can be exerted on a charge.

Estimate: $PE = 10^{-2}\,K \sim 10^{-25}\,J$

What potential change does this correspond to?

$PE = Q \cdot \Delta V = 10^{-19}\,C \cdot \Delta V \Rightarrow \Delta V = 10^{-6}\,V$

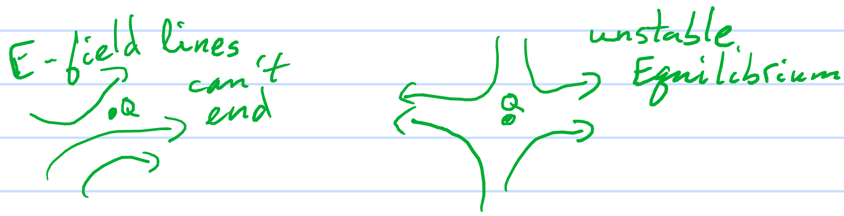
Even for small dipole traps $l \sim 10^{-5}\,m \Rightarrow E = 10^{-1}\,\frac{V}{m} = 1\,\frac{mV}{cm}$

$\Rightarrow$ Small electric field

Two typical methods used: Paul Trap (only E-fields) and Penning traps (E & B-fields)

Paul Traps get around the problem from Earnshaw's Theorem: There is no stable equilibrium for a charge in electrostatic fields where the external charge density is zero.

Explanation 1:



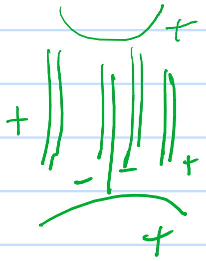
Explanation 2: Put small sphere around Q

Do Gauss's law, but not for field from Q $\oint \vec{E} \cdot d\vec{a} = 0$

This implies there are some spots where E points in (restoring force) <u>and</u> some spots where E points out (destabilizing).

The trick of the Paul trap is to use time varying fields to get dynamic stabilization.

First, we need to see how the equations of motion are obtained. Look at some Paul traps.



The two end caps have static potentials. Near the center, they give a potential of the form

$$V \cong C\left(z^2 - \frac{x^2+y^2}{2}\right)$$

lowest order in x, y, z with correct $z \leftrightarrow -z$ and cylindrical symm.

The rods give $V \cong \tilde{C}\,(x^2 - y^2)$ lowest order in x, y, z with no z dep and $x \leftrightarrow y$ change sign

Paul's insight is to have the rods with oscillating voltage

A common parameterization has the PE be

$$PE = Q K V_{end} \frac{z^2 - \frac{x^2+y^2}{2}}{z_0^2} - Q V_{RF} \frac{x^2 - y^2}{r_0^2} \cos(\Omega_{RF} t)$$

The equations of motion are from Newton

$$\ddot{z} = -\frac{2QKV_{end}}{m z_0^2} z$$

$$\ddot{x} = \frac{QKV_{end}}{m z_0^2} x + \frac{2QV_{RF}}{m r_0^2} x \cos(\Omega_{RF} t)$$

$$\ddot{y} = \frac{QKV_{end}}{m z_0^2} y - \frac{2QV_{RF}}{m r_0^2} y \cos(\Omega_{RF} t)$$

For historical reasons, the time is scaled as $t = \frac{2}{\Omega_{RF}} \tau$

The equations all have the same form.

$$\frac{d^2 u_i}{d\tau^2} + \left[a_i - 2 q_i \cos(2\tau)\right] u_i = 0$$ Mathieu equation

$$a_x = a_y = -\frac{1}{2} a_z = -K \frac{4 Q V_{end}}{m \Omega_{RF}^2 z_0^2}$$

$$q_z = 0, \quad q_x = -q_y = \frac{2 Q V_{RF}}{m \Omega_{RF}^2 r_0^2}$$

The equation for z is simple harmonic oscillator.

$$z(t) = z(0) \cos(\omega_z t) + \frac{V_z(0)}{\omega_z} \sin(\omega_z t) \qquad \omega_z^2 = K \frac{2 Q V_{end}}{m z_0^2}$$

There's a wide range of values. I was doing calculations for
$K = 0.22$, $V_{RF} = 85$ V, $z_0 = 2$ mm, $M_{Ca^+} = 40$ amu
$\omega_z = 4.7 \times 10^6$ rad/s $\quad f_z = 750$ kHz

How to think about the trapping in the other directions?

Look at plots of motion for different a, q.
Look at stability diagram.

How to think about trapping? Go back to motion in uniform E-field.

$$\dot{v} = \frac{QE}{m}\cos(\omega t) \Rightarrow v = \frac{QE}{m\omega}\sin(\omega t)$$

The cycle average $\langle \tfrac{1}{2} m v^2 \rangle = \tfrac{1}{2} m \frac{Q^2E^2}{m^2\omega^2}\langle \sin^2(\omega t)\rangle = \frac{Q^2E^2}{4m\omega^2}$

In the Paul trap $E = \frac{V}{L^2} x \cos(\omega t)$
An effective PE

$$PE_{eff} = \frac{1}{2}\left(\frac{Q^2E^2}{4m\omega^2}\right) = \frac{1}{2}\,\frac{Q^2V^2}{2L^4 m\omega^2}\,x^2$$ Attractive harmonic osc.

If the $PE_{eff} > PE_{end}$, then get trapping. Look at trapping figure.

How to explain the instability at large q? Oscillation from PE_{eff} becomes comparable to Ω_{RF}.

Since the force is periodic, can look for solutions of the Floquet form.

$$u = A e^{i\omega_u t}\Big[1 + \alpha_2\cos(2\tau) + \alpha_4\cos(4\tau) + \ldots + \beta_2\sin(2\tau) + \beta_4\sin(4\tau) + \ldots\Big]$$

I will not go through how to get the equations for the α, β and ω_u. Will only look at the case where the α, β are small. α_2, β_2 are larger than α_4, β_4 ...

Looking for approximate solutions when a and q are small. This gives ω_u, α_2, β_2 small. So ignore terms that are products of these.

$$\ddot{u} \cong -\omega_u^2 A e^{i\omega_u \tau} - 4 A e^{i\omega_u \tau} \alpha_2 \cos(2\tau)$$

$$[a - 2q\cos(2\tau)]\, u \cong A e^{i\omega_u \tau} \left[a - 2q\cos(2\tau) - 2q\alpha_2 \tfrac{1}{2}\right]$$

$$\omega_u^2 = a - q\alpha_2 \qquad -4\alpha_2 = +2q \Rightarrow \alpha_2 = -q/2$$

$$\omega_u = \sqrt{a_u + \tfrac{1}{2} q_u^2}$$

The actual frequency of motion in x + y need to put back the $\Omega_{RF}/2$

$$\omega_{x,y} \cong \sqrt{a_{x,y} + \tfrac{1}{2} q_{x,y}^2} \; \frac{\Omega_{RF}}{2}$$

The x, y have negative a $\Rightarrow$ One limit of the stable motion is
$q_i \gtrsim \sqrt{-2a_{x,y}}$

Why is there an upper limit on q for stability? ω_u needs to be much less than 1

What about cooling of ions? People have tried putting the ion in a cold neutral gas.
1) Unlike time independent potentials ions don't go to Maxwell-Boltzmann unless $M_{ion} \gg M_{neut}$. The $T_{ion} > T_{neut}$ with runaway heating for $M_{ion} \gtrsim M_{neut}$.
2) Recombination at high neutral density $\Rightarrow$ Bound (ion + neutral)

Laser cooling often can be easily made to work.

Ca^+ 4s–4p $\frac{\hbar\Gamma}{2} = 0.53\,\text{mK}\, k_B \Rightarrow$ optical molasses $T \sim 530\,\mu\text{K}$
$\Gamma = 1.39\times10^8$ rad/s linewidth larger than Ω_{RF}

Does the Paul trap work for more than 1 ion?

Many ions of same type can make crystal. See review.
See transition line → buckled line → ...
Ions with nearly the same mass can be sympathetically cooled.
For example Ca^+ $M=40$ SiO^+ $M=44$

The effective PE is inversely proportional to mass ⇒ small mass ions are tight and larger mass on outside. See images

Some ions with several isotopes, laser only cools one. If only 1 laser for laser cooling, then flourescing ions pushed away.

Do the ions stay cold? If the ions are very cold (crystal), then they do not heat up even if not in a line. If the ions are hot, they will further heat up. Why? They feel the RF with same phase and tend to move together. When the ions are in a crystal can do a harmonic approx around each position. When ions are hotter, then have non harmonic interactions.

The Penning Trap $\vec{B} = (0, 0, B)$ $PE = QKV_{end}\dfrac{z^2 - \frac{x^2+y^2}{2}}{z_0^2}$

The magnetic field prevents the ion from going out in x, y and the end caps prevents escape in z.

To understand the motion, look at simpler case with $\vec{E}$ and $\vec{B}$ both uniform $\vec{F} = Q[\vec{E} + \vec{v} \times \vec{B}]$

$$\dot{V}_x = \frac{QE_x}{m} + \frac{QB}{m}V_y = a_x + \omega_c V_y \qquad \dot{V}_y = \frac{QE_y}{m} - \frac{QB}{m}V_x = a_y - \omega_c V_x$$

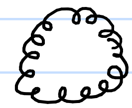
$$V_y = -\frac{a_x}{\omega_c} + V_c\cos(\omega_c t + \phi) = -E_x/B + V_c\cos(\omega_c t + \phi) \qquad V_x = \frac{a_y}{\omega_c} + V_c\sin(\omega_c t + \phi) = E_y/B + V_c\sin(\omega_c t + \phi)$$

The motion is the usual cyclotron motion plus a constant drift velocity

$\vec{V}_{drift} = (\vec{E} \times \vec{B})/B^2$ $\quad\perp$ to $\vec{E}$ and $\vec{B}$

Large B => small V_{drift}

The Penning trap motion is approximately



$$\vec{E} = \frac{K V_{end}}{z_0^2}(x, y, -2z)$$

$$(\vec{E} \times \vec{B})/B^2 = \frac{K V_{end}}{z_0^2 B}(y, -x, 0) = \vec{V}_{drift}$$

Move in a circle

$$\dot{x} \simeq \frac{K V_{end}}{z_0^2 B} y \qquad \dot{y} = -\frac{K V_{end}}{z_0^2 B} x$$

$$x = A\sin(\omega_d t + \phi) \qquad y = A\cos(\omega_d t + \phi)$$

$$\omega_d = \frac{K V_{end}}{z_0^2 B}$$

Can you have $\omega_d > \omega_c$???

Expect not to work when $\omega_c \sim \omega_d$ $\quad \frac{QB}{m} \sim \frac{K V_{end}}{z_0^2 B}$

This sets a lower limit $B^2 \gtrsim \frac{m}{Q}\frac{K V_{end}}{z_0^2}$

Now do the actual equations of motion

$$\dot{x} = V_x \qquad \dot{V}_x = \frac{QK V_{end}}{z_0^2 m} x + \omega_c V_y = \omega_d \omega_c x + \omega_c V_y$$

$$\dot{y} = V_y \qquad \dot{V}_y = \frac{QK V_{end}}{z_0^2 m} y - \omega_c V_x = \omega_d \omega_c y - \omega_c V_x$$

These are 4 coupled, linear differential equations. Look for the 4 eigenmodes.

$$x = x_0 e^{i\omega t} \qquad y = y_0 e^{i\omega t} \qquad V_x = V_{x0} e^{i\omega t} \qquad V_y = V_{y0} e^{i\omega t}$$

This gives 4 linear equations

$$i\omega\, x_0 = V_{x0} \qquad i\omega\, V_{x0} = \omega_d \omega_c x_0 + \omega_c V_{y0}$$
$$i\omega\, y_0 = V_{y0} \qquad i\omega\, V_{y0} = \omega_d \omega_c y_0 - \omega_c V_{x0}$$

Substitute to get 2 equations

$$-\omega^2 x_0 = \omega_d \omega_c x_0 + i\omega\omega_c y_0 \Rightarrow (\omega^2 + \omega_d\omega_c) x_0 + i\omega\omega_c y_0 = 0$$
$$-\omega^2 y_0 = \omega_d \omega_c y_0 - i\omega\omega_c x_0 \Rightarrow -i\omega\omega_c x_0 + (\omega^2 + \omega_d\omega_c) y_0 = 0$$

The determinant is $(\omega^2 + \omega_d\omega_c)^2 - \omega^2\omega_c^2 = 0$

$$\omega^2 + \omega_d\omega_c = \pm\omega\omega_c \Rightarrow \omega^2 \mp \omega\omega_c + \omega_d\omega_c = 0$$

The frequencies are

$$\omega = \frac{\pm\omega_c \pm (\omega_c^2 - 4\omega_d\omega_c)^{1/2}}{2}$$

The 4 signs are independent

$$= \pm\frac{1}{2}\Big[\omega_c \overset{\text{indep}}{\pm} \omega_c \left(1 - \tfrac{4\omega_d}{\omega_c}\right)^{1/2}\Big]$$

If $\omega_d > \omega_c/4$ then ω is imaginary $\Rightarrow$ not stable

If $\omega_d \ll \frac{\omega_c}{4}$

$$\omega = \pm\frac{1}{2}\Big[\omega_c \pm \omega_c \left(1 - \frac{2\omega_d}{\omega_c}\right)\Big]$$
$$= \pm(\omega_c - \omega_d) \quad \text{and} \quad \pm\omega_d$$

Essentially cyclotron motion

Essentially the drift motion

When you have many ions in a Penning trap, the E-field from the other ions add to the ω_d. When the ions are cold you get the whole plasma rotating like a solid body. To derive the rotational frequency use $E_z \sim 0$, $E_{x,y} = (x,y)\, C$

$$\vec{\nabla}\cdot\vec{E} = 2C = \frac{Qn}{\varepsilon_0}$$

n = number density

$$C = \frac{Qn}{2\varepsilon_0}$$

Now use the approximation for drift velocity

$$V_d \approx E/B = \frac{Qn}{2\varepsilon_0 B} r$$

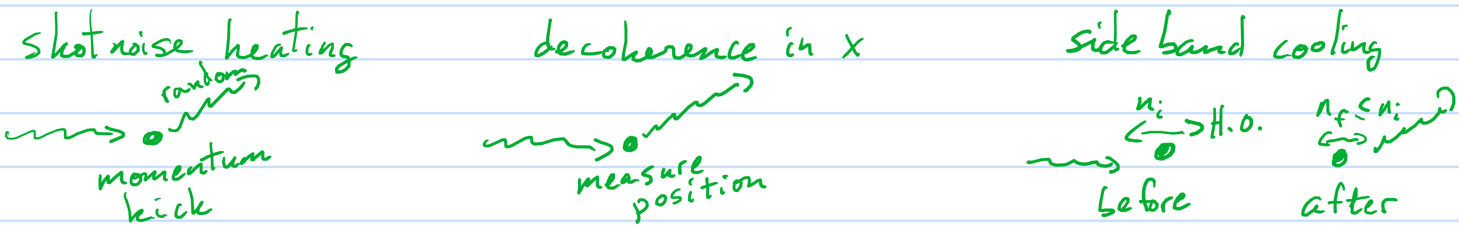
Since V_d is proportional to r, the system behaves like a rotating solid body

$$\omega_d \sim V_d/r = \frac{Qn}{2\varepsilon_0 B}$$

Notice the trends in ω_d. Look at pictures from Wineland, ...

One subtle effect is centrifugal separation. The next term (we neglected) gives preference for larger mass ions to be at the edge of the plasma.

Now examine the treatment of shot noise heating, decoherence in center of mass coords, and side band cooling. All three depend on light having a wavelength/momentum.



Have the Hamiltonian of the "atom" with both the center of mass and internal degree's of freedom. For simplicity, will imagine C.O.M. is a harmonic oscillator <u>and</u> the light can only go in $\pm x$ direction.

<u>Side band cooling</u>

The atomic state can be written as $|n_g, \psi_g\rangle$ or $|n_e, \psi_e\rangle$ where the n is the H.O. quantum number and g, e are the ground or excited electronic state.

The atom energy is the sum

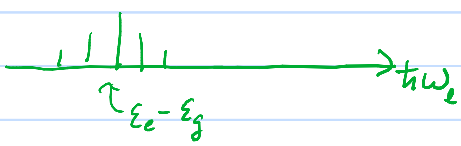
$$E_{n,g} = \hbar\omega_{HO} n_g + \varepsilon_g \qquad \text{and} \qquad E_{n,e} = \hbar\omega_{HO} n_e + \varepsilon_e$$

Suppose the lifetime of the excited state is long enough so that

$$\omega_{HO} \gg \Gamma_e$$

The absorption spectrum gives peaks at the energies

$$\hbar\omega_{light} = \varepsilon_e - \varepsilon_g + \hbar\omega_{HO}(n_e - n_g)$$



For long wavelength light, only 3 strong peaks $n_e - n_g = 0, \pm 1$

Before going through the math, note a couple important situations: 1) if $n_g = 0$, there can't be $n_e - n_g = -1$. 2) the ratio of peak heights gives info on the temperature. 3) In both absorption and emission, the $\Delta n = 0$ peak is strongest. 4) Cooling scheme to have $\hbar\omega_\ell = \varepsilon_e - \varepsilon_g - \hbar\omega_{HO}$ (Why works?)

The following treatment is not quite right because we're in the wrong gauge but gives OK results for near resonant transitions.

Back in Chap 7 we had for 2 states:

$$i\hbar\dot{a}_1 = E_1 a_1 + \hbar\Omega\frac{E(t)}{E_0}a_2 \qquad i\hbar\dot{a}_2 = E_2 a_2 + \hbar\Omega\frac{E(t)}{E_0}a_1$$

with $\qquad \Psi(t) = |\psi_1\rangle a_1 + |\psi_2\rangle a_2$

First decide what the wave function should look like

$$\Psi(t) = \sum_{n_1} |n_1, \psi_1\rangle a_{1,n_1} + \sum_{n_2} |n_2, \psi_2\rangle a_{2,n_2}$$

Substitute into Schrodinger equation and project the basis functions

$$i\hbar \dot{a}_{1,n_1} = (E_1 + \hbar\omega_{HO} n_1)\, a_{1,n_1} + \hbar\Omega \sum_{n_2} \frac{\langle n_1 | E(t) | n_2\rangle}{E_0} a_{2,n_2}$$

$$i\hbar \dot{a}_{2,n_2} = (E_2 + \hbar\omega_{HO} n_2)\, a_{2,n_2} + \hbar\Omega^* \sum_{n_1} \frac{\langle n_2 | E(t) | n_1\rangle}{E_0} a_{1,n_1}$$

Now need to figure out what the $\langle n | E(t) | n'\rangle$ has to be.

Chap 7: $E(t)/E_0 = \cos(\omega t) \Rightarrow \langle n_1 | E(t) | n_2\rangle = \delta_{n_1,n_2} \cos(\omega t)$

Work out the case for laser with $\vec{k} = (k,0,0)$

$$\langle n_1 | \cos(\omega t - kX) | n_2\rangle \neq \delta_{n_1,n_2} \cos(\omega t)$$

Evaluate the Taylor series expansion of the E-field in x

$$e^{i\omega t} e^{-ikx} = e^{i\omega t}\left(1 - ikx - \frac{k^2}{2}x^2 + i\frac{k^3}{6}x^3 \ldots\right)$$

Evaluate using the raising/lowering operators:

$$X = \sqrt{\frac{\hbar}{2m\omega_{HO}}}\,(a^+ + a) \qquad a^+\psi_n = \sqrt{n+1}\,\psi_{n+1} \qquad a\,\psi_n = \sqrt{n}\,\psi_{n-1}$$

Supposing the quantum length scale of harmonic oscillator is much smaller than the wavelength of light, $\sqrt{\frac{\hbar}{2m\omega_{HO}}}\, k \ll 1$, then only need to keep the first two terms.

$$\langle n_1 | e^{i\omega t} e^{-ikx} | n_2\rangle = e^{i\omega t}\left[\delta_{n_1,n_2} - ik\sqrt{\frac{\hbar}{2m\omega_{HO}}}\left(\sqrt{n_1}\,\delta_{n_1,n_2+1} + \sqrt{n_2}\,\delta_{n_1+1,n_2}\right)\right]$$

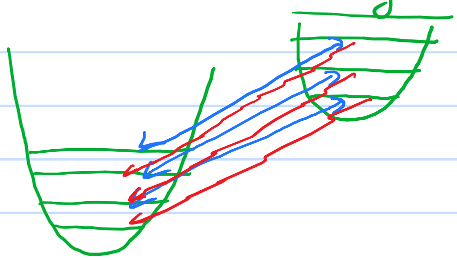
The next term (which is dropped), is proportional to

$$k^2\langle n_1 | X^2 | n_2\rangle \propto k^2 \frac{\hbar}{2m\omega_{HO}}\Big[\underset{\uparrow\text{ correct the } \delta n=0 \text{ rate}}{\delta_{n_1 n_2}} + \underbrace{(\,)\,\delta_{n_1,n_2+2} + (\,)\,\delta_{n_1+2,n_2}}_{\delta n=2 \text{ transitions}}\Big]$$

This is enough to get idea of side band cooling.
The transition rate for $g \to e$ with $n_e = n_g$ is nearly the same for absorption and spontaneous emission)
The transition rate for $g \to e$ with $n_e = n_g - 1$ is smaller by the factor of $k^2 \frac{\hbar}{2m\omega_{HO}} n_e$ (also for spontaneous emission).

Similar for $n_e = n_g + 1$

How sideband cooling works



Drive the $\delta n = -1$ transition
Spontaneous emission mainly $\delta n = 0$
Sideband cooling comes from the momentum kick from the photon

Describe Dicke Narrowing: The trapping potential at $x=0$ and standing wave of light of form $\cos(\omega t)\cos(kx)$ Strongly reduces the Doppler broadening and line width if $\omega \gg \Gamma$

Quantum description of shot noise heating must also be based on the momentum kick from photon. Now we will be in the limit of $\Gamma \gg \omega_{HO}$ (decay happens quickly on time scale of oscillator period).

Classical treatment no restoring force: The momentum gets a series of kicks $\delta p_1, \delta p_2, \ldots$ at times $t_1, t_2, \ldots$

$p_f = p_i + \delta p_1 + \delta p_2 \ldots \delta p_N$ is the final momentum.

$\Delta E = \frac{p_f^2}{2m} - \frac{p_i^2}{2m}$ is the change in energy

$$\Delta E = \frac{1}{m} p_i (\delta p_1 + \delta p_2 + \ldots \delta p_N) + \frac{1}{2m}(\delta p_1 + \delta p_2 + \ldots \delta p_N)(\delta p_1 + \delta p_2 + \ldots \delta p_N)$$

The δp_j are random. All of the cross terms can be dropped after many kicks because they average to 0

$$\Delta E = \frac{1}{2m}(\delta p_1^2 + \delta p_2^2 + \ldots \delta p_N^2) = \frac{N\langle \delta p^2 \rangle}{2m}$$

The energy (on average) increases linearly with number of kicks.

$\frac{\Delta E}{t_N}$ = Shot noise heating rate $\equiv \dot{E} = \frac{\langle \delta p^2 \rangle}{2m}\left(\frac{t_f}{N}\right)^{-1} = \frac{\langle \delta p^2 \rangle}{2m}\Gamma_{kick}$

This should make sense because it is the kick rate times the average energy gain per kick.

Can also think of this in terms of an ensemble average. After 1 kick, The momentum is

$P = P_i + \delta P$

The change in energy is

$\Delta E = P_i \delta P/2m + \delta P^2/2m$

Now ensemble average

$\langle \Delta E \rangle = P_i \langle \delta P \rangle/2m + \langle \delta P^2 \rangle/2m$

Now account for average time between kicks

$\dot{E} = \langle \Delta E\, \Gamma \rangle = [\langle \delta P^2 \rangle/2m]\Gamma_{sc}$

Before doing quantum version, note that $\langle \delta P^2 \rangle/2m > \hbar\omega_{ho}$ is not very interesting. The opposite limit $\langle \delta P^2 \rangle/2m \ll \hbar\omega_{ho}$ gives insight into quantum limit. Before doing the theory, ask whether we should expect heating in this limit.

For the first derivation, will start the "atom" in a particular vibrational state n. After getting the results for the simplified case, will examine the case where the atom is in a mixed state.

Before the kick $\psi_i(x) = \psi_n(x)$. What is ψ after a momentum kick δP

$$\psi_f(x) = e^{i\delta p x/\hbar}\psi_i(x) = e^{i\delta p x/\hbar}\psi_n(x)$$

Taylor series expand the exponential.

$$\psi_f(x) = \left(1 + i\,\delta p\, x/\hbar - \frac{\delta p^2}{2\hbar^2}x^2 \ldots\right)\psi_n(x)$$

$$= \psi_n(x) + \frac{i\delta P}{\hbar}x\psi_n(x) - \frac{\delta P^2}{2\hbar^2}x^2\psi_n(x) \ldots$$

$$\psi_f(x) = \psi_n(x) + \frac{i\delta p}{\hbar}\sqrt{\frac{\hbar}{2m\omega}}(a+a^\dagger)\psi_n(x) - \frac{\delta p^2}{2\hbar^2}\frac{\hbar}{2m\omega}(aa + aa^\dagger + a^\dagger a + a^\dagger a^\dagger)\psi_n(x) \ldots$$

$$= \psi_n(x) + \frac{i\delta p}{\hbar}\sqrt{\frac{\hbar}{2m\omega}}\left[\sqrt{n}\,\psi_{n-1} + \sqrt{n+1}\,\psi_{n+1}\right] - \frac{\delta p^2}{2\hbar^2}\frac{\hbar}{2m\omega}\left(\sqrt{n(n-1)}\,\psi_{n-2} + (2n+1)\psi_n + \sqrt{(n+1)(n+2)}\,\psi_{n+2}\right) \ldots$$

Now need to compute the final energy

$$E_f = \sum_n n\hbar\omega\, |C_n|^2$$

$$= n\hbar\omega\left(1 - \frac{\delta p^2}{2\hbar m\omega}(n+1/2)\right)^2 + (n-1)\hbar\omega\frac{\delta p^2}{2m\hbar\omega}n + (n+1)\hbar\omega\frac{\delta p^2}{2m\hbar\omega}(n+1)$$

$$= n\hbar\omega - \frac{\delta p^2}{2m}2n(n+1/2) + \frac{\delta p^2}{2m}\left[n(n-1) + (n+1)^2\right]$$

$$= n\hbar\omega + \frac{\delta p^2}{2m}$$ Woo!

This gives exactly the same result as the classical derivation

$\Delta E = \delta p^2/2m$ $\qquad\qquad$ $\dot{E} = \frac{\langle \delta p^2 \rangle}{2m}\Gamma_{sc}$

It doesn't matter that $\delta p^2/2m \ll \hbar\omega$

The derivation when $\psi_i = C_0\psi_0 + C_1\psi_1 + \cdots$ proceeds in the same way except there will be terms first order in δp that comes from ψ_n getting $n\pm1$ character that overlaps with $\psi_{n\pm1}$. But just like the classical case, those terms average to 0.

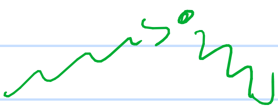
Decoherence: The quantum treatment above misses out on an important consideration: the photons that scatter from the atom become entangled with it. By not measuring the scattered photons, the wave function picture is impossible. We need to go back to the density matrix picture.

From Chap 7, the Hamiltonian part of the interaction gives:

$$\frac{\partial \rho}{\partial t} = \frac{1}{i\hbar}[H, \rho] + \text{effects from photon scattering}$$

The harmonic oscillator can be treated as $H = \hbar\omega\, a^{+}a$

How to think what the effects from photon should be?



Scattering photon should give localization of atom with size scale $\delta x \sim 1/k \sim \hbar/p_{ph}$

Within the density matrix $\rho(x, x'; t)$ should get smaller as $|x - x'|$ gets larger

$$\frac{\partial \rho(x,x',t)}{\partial t} = \frac{1}{i\hbar}[H, \rho] - \underbrace{\overset{\text{const}}{\Lambda} \cdot (x-x')^2 \rho(x, x', t)}_{\text{how to get this?}}$$

Remember how the density matrix is defined for pure states

$$\rho = |\Psi\rangle\langle\Psi|$$

$$\rho(x, x') = \langle x|\Psi\rangle\langle\Psi|x'\rangle = \Psi(x)\,\Psi^{*}(x')$$

After a photon scattering, there is a random δp

Photon term $\rho(x,x',t+\delta t) = \left\langle e^{i\delta p x/\hbar}\, \rho(x, x'; t)\, e^{-i\delta p x'/\hbar} \right\rangle_{\delta p}$ average over δp

$$= \cancelto{1}{\langle \delta p^{0}\rangle}\, \rho(x,x',t) + \frac{i}{\hbar}\cancelto{0}{\langle \delta p\rangle}\,(x-x')\,\rho(x,x',t)$$

$$- \frac{1}{2\hbar^2}\langle \delta p^2\rangle (x-x')^2 \rho(x,x',t)$$

$$\frac{\rho(x,x',t+\delta t) - \rho(x,x',t)}{\delta t} = -\underbrace{\overset{=\Gamma_{sc}}{\boxed{\frac{1}{\delta t}}}\,\frac{1}{2\hbar^2}\langle \delta p^2\rangle}_{\Lambda}(x - x')\,\rho(x,x',t)$$

This is the form advertised above.
Is there a connection to shot noise heating?

To find the rate that the energy is increasing use the standard formula for operators

$$\frac{d}{dt}\langle A\rangle = \mathrm{Tr}\left(A\frac{\partial\rho}{\partial t}\right) + \mathrm{Tr}\left(\frac{\partial A}{\partial t}\rho\right)$$

$$\frac{dE}{dt} = \frac{d}{dt}\langle H\rangle = \mathrm{Tr}\left(H\frac{\partial\rho}{\partial t}\right) = \frac{1}{i\hbar}\mathrm{Tr}\big(\overbrace{H\rho H - H H\rho}^{H[\rho,H]}\big) - \Lambda\,\mathrm{Tr}\left(H(x-x')^2\rho\right)$$

$$= \frac{1}{i\hbar}\underbrace{\cancel{\mathrm{Tr}\left(H H\rho - H H\rho\right)}}_{\to 0} - \Lambda\,\mathrm{Tr}\left(-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2}(x-x')^2\rho(x,x',t)\right)$$

$$= \Lambda\frac{\hbar^2}{2m}\mathrm{Tr}\left(2\rho + 4(x-x')\frac{\partial\rho}{\partial x} + (x-x')^2\frac{\partial^2\rho}{\partial x^2}\right)$$

$$= \Lambda\frac{\hbar^2}{m} = \Gamma_{sc}\frac{1}{2\hbar^2}\langle\delta p^2\rangle\,\hbar^2/m$$

$$\frac{dE}{dt} = \Gamma_{sc}\frac{\langle\delta p^2\rangle}{2m}$$

This is the same answer as before! Shot noise heating and localization and spatial decoherence are different aspects of the same physical situation.

What about "extensions" of Q.M. that lead to the localization of micro (or larger) scale objects?

Mention paper that uses G.R. to get spatial decoherence?