# PHYS461, Test 2, Spring 2018

*You must show work to get credit. There are integrals at the back of the book and at the bottom of the page that might be useful.*

**For any time dependent perturbation theory problems, only solve through 1$^{\mathrm{st}}$ order.**

(1) (5 pts) Estimate the lowest few rotational energy levels for a silica nanosphere of radius 50 nm. Give your answer in K.

(2) (5 pts) A particle of mass $M$ experiences a smooth potential $V(x)$ which is 0 for $|x| > L$ and has $V(x) < E$ for all x. For $x < -L$, $\psi(x) = A\exp(-ikx)$ where $A$ is real. At the WKB level, determine the wave function for all $x$. Get all amplitudes and phases correct.

(3) (5 pts) A spin-1 system has the Hamiltonian $H = \Delta E S_z/\hbar + \exp(-t^2/T^2)V_0 S_x/\hbar$ with small $V_0$. The system starts in spin $-1$ for $t \ll -T$. Determine the probability that it is in the spin 0 or the spin 1 states at $t \gg T$. (For the spin matrices see pg 195 Prob 4.53 [all of the *non-zero* off-diagonal elements for spin-1 $S_x$ are $\hbar/\sqrt{2}$]; see board).

(4) (5 pts) A particle of mass $M$ is in a potential $V(x) = \infty$ for $x < 0$, $V(x) = 0$ for $0 < x < L$, and $V(x) = V_0 - (x - L)F$ for $x > L$ where $V_0, F$ are positive constants. The particle starts in the state trapped between 0 and $x \sim L$ with energy $E_0 < V_0$. As accurately as possible, estimate the lifetime of this state.

(5) (10 pts) A spin-1/2 particle has the Hamiltonian $H = (\Delta E/2)(t/T)\sigma_z$. (a) Obtain the two time dependent eigenstate solutions for this system. (b) The term $V_0\sigma_x \exp(-\alpha t^2)$ is added to $H$ where $V_0$ is a small constant. If the system starts in the $-1/2$ state at $t \to -\infty$, what is the probability it will be in the $+1/2$ state at $t \to \infty$? (Hint: for part (b), you can't directly use the perturbation theory in the book but appropriately modifying Eq. [9.6] will give you the answer in a couple steps.)

(6) (10 pts) (a) As accurately as possible, determine the eigenergies, $E_n$, for a particle of mass M in a potential which is $V(x) = Fx$ for $x > 0$ and $\infty$ for $x < 0$. (b) Give the wave function in the classically allowed and unallowed regions (don't worry about normalization). (c) The wave function is $(|\psi_n\rangle + i|\psi_{n+1}\rangle)/\sqrt{2}$ at $t = 0$. What is the smallest time where the expectation value of any operator is the same as at $t = 0$. (d) In the limit of $n \gg 1$, evaluate this expression for the quantum period.

(7) (10 pts) An *electron* has the potential energy $V(x, y) = (1/2)M\omega^2(x^2 + y^2)$ and is in a laser field. The laser field gives: $\Delta V = \cos(\omega_l t)\exp(-t^2/T^2)A_l(p_x + kyp_x)$ where $\omega_l = 2\pi f_l$ with $f_l$ the frequency of the light, $k = 2\pi/\lambda$ with $\lambda$ the wavelength of the light, and $A_l$ is a small constant. Take $\omega = 2\pi 10^{15}$ Hz. The electron starts in the ground state and the laser is weak. (a) Give the energy spacing of the states in eV. (b) What are the only states that the laser can cause a transition into with probability proportional to $|A_l|^2$? (c) What is the resonance $f_l$ needed for each of the allowed cases? (d) Calculate the probability for each transition versus $\omega_l$. (e) What is the *relative probability* for each transition when that transition is perfectly on resonance? Given numerical values.

Possibly useful integral $\int_{-\infty}^{\infty} \exp(iax)\exp(-x^2)dx = \sqrt{\pi}\exp(-a^2/4)$

Prob 1 The eigenenergies are determined from a Hamiltonian like $H = L^2/2I$ This gives energies $\mathcal{E}_\ell = (\hbar^2/2I)\,\ell(\ell+1)$ $= 0,\ 2,\ 6,\ 12 \ldots\ \hbar^2/2I$

You need to estimate the moment of inertia. For a shell the $I = MR^2$. Since the average mass is at distances less than $R$, $I \sim \frac{1}{2} MR^2$ (you should look up the actual value!)

The mass can be found from $M = \frac{4}{3}\pi R^3 \rho \sim 4R^3\rho$

For density you know that sand sinks in water but is a lot less dense than lead $\rho \sim 3\times10^3\ \mathrm{kg/m^3}$ (you should look up the actual value!)

$$I \sim 2R^5\rho \sim 2\,(50\times10^{-9}\,\mathrm{m})^5\ 3\times10^3\ \mathrm{kg/m^3} \sim 2\times10^{-33}\ \mathrm{kg\,m^2}$$

$$\hbar^2/2I \sim \frac{(10^{-34}\ \mathrm{Js})^2}{2\cdot 2\times10^{-33}\ \mathrm{kg\,m^2}} \sim 2.5\times10^{-36}\ \mathrm{J} \sim 2\times10^{-13}\ \mathrm{K}$$

$$\mathcal{E} \sim 0,\ 4\times10^{-13}\ \mathrm{K},\ 12\times10^{-13}\ \mathrm{K},\ 24\times10^{-13}\ \mathrm{K},\ \ldots$$

Prob 2 From Eq. 8.10 $\psi(x) \cong \frac{C}{\sqrt{p(x)}}\, e^{\pm i\int^x p(x')dx'/\hbar}$

To get the amplitude right $C/\sqrt{p(L)} = A$ $\boxed{C = \sqrt{2mE}\,A}$

To get the sign of the rate of phase change you must use only the $-$ sign

To get the phase right $kL = \int_{-L}^{x} p(x')dx'/\hbar + \varphi(-L) \Rightarrow \varphi(-L) = kL$

Put it all together $\boxed{\psi(x) = A\sqrt{\frac{2mE}{p(x)}}\ e^{-i\int_{-L}^{x} p(x')dx'/\hbar}\ e^{ikL}}$

Prob 3 Use 1st order time dependent perturbation theory

The energies are $E_s = s\,\Delta E$

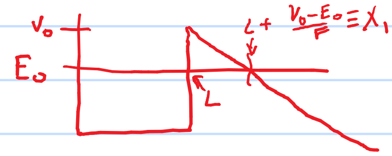
Only the coupling $s=-1$ to $s=0$ is nonzero. This means $\boxed{P_1 = 0}$

The coupling to 0 is $H_{ab} = \frac{V_0}{\sqrt{2}} e^{-t^2/T^2}$. Use Eq. [9.17]

$$C_0^{(1)}(\infty) = \frac{1}{i\hbar}\int_{-\infty}^{\infty} \frac{V_0}{\sqrt{2}} e^{-t^2/T^2} e^{i\Delta E t/\hbar}\,dt = \frac{V_0\sqrt{\pi}}{i\hbar\sqrt{2}}\, T\, e^{-T^2\Delta E^2/4\hbar^2}$$

$$P_0 = |C_0^{(1)}(\infty)|^2 = \frac{|V_0|^2\pi}{2\hbar^2}T^2 e^{-T^2\Delta E^2/2\hbar^2}$$

Prob 4 This is Example 8.2 but with




Use Eq. [8.28]. The pre factor is

$2L/v = 2L/\sqrt{2E_0/m}$

$$\gamma = \frac{1}{\hbar}\int_L^{X_1} |p(x')|\,dx' = \frac{1}{\hbar}\int_L^{X_1} \sqrt{2m}\sqrt{V_0 - E_0 + LF - xF}\,dx = \frac{1}{\hbar}\sqrt{2m}\left(-\frac{2}{3F}\right)\left(V_0 - E_0 + LF - XF\right)^{3/2}\Big|_L^{X_1}$$

$$= \frac{1}{\hbar}\sqrt{2m}\,\frac{2}{3F}(V_0 - E_0)^{3/2}$$

Put it all together $\boxed{\tau = 2L\sqrt{\frac{m}{2E_0}}\, e^{-\sqrt{2m}\,4(V_0-E_0)^{3/2}/(3F\hbar)}}$

Prob 5 The equations for the unperturbed H is

$$i\hbar\dot{C}_\pm = \pm\frac{\Delta E}{2}\frac{t}{T}C_\pm \Rightarrow \frac{dC_\pm}{C_\pm} = \mp\frac{i\Delta E}{\hbar 2}\frac{t}{T}dt \qquad C_\pm(t) = C_\pm(0)\,e^{\mp\frac{i\Delta E t^2}{4\hbar T}}$$

$$|\Psi(t)\rangle = C_+(t)\,e^{-i\Delta E t^2/4T\hbar}|+\tfrac{1}{2}\rangle + C_-(t)\,e^{+i\Delta E t^2/4T\hbar}|-\tfrac{1}{2}\rangle$$ Modified Eq [9.6]

Now modify Eq [9.17] $\frac{dC_+}{dt} = \frac{1}{i\hbar}V_0 e^{-\alpha t^2} e^{i\Delta E t^2/2T\hbar}$

Integrate both sides $C_+(\infty) = \frac{V_0}{i\hbar}\int_{-\infty}^{\infty} e^{-(\alpha - \frac{i\Delta E}{2T\hbar})t^2}dt = \frac{V_0}{i\hbar}\frac{1}{\sqrt{\alpha - \frac{i\Delta E}{2T\hbar}}}$

Compute the probability $P_+ = \frac{\pi|V_0|^2}{\hbar^2}\frac{1}{\sqrt{\alpha^2 + (\frac{\Delta E}{2T\hbar})^2}}$

Not asked for but an interesting point: There is well defined limit when $\alpha \to 0$ $P_+ = 2\pi|V_0|^2 T/(\hbar|\Delta E|)$ : The diabatic limit of Landau-Zener crossing.

Prob 6 This is the case covered by Eq. [8.47] $\int_0^{X_2} p(x)\,dx = (n - \frac{1}{4})\pi\hbar$

$X_2 = E/F$ and $p(x) = \sqrt{2m}\,(E - Fx)^{1/2}$ Insert into equation

$$\sqrt{2m}\int_0^{E/F}(E - Fx)^{1/2}dx = \sqrt{2m}\left(-\frac{2}{3F}\right)(E - Fx)^{3/2}\Big|_0^{E/F} = \sqrt{2m}\left(\frac{2}{3F}\right)E_n^{3/2} \Rightarrow \boxed{E_n = \frac{1}{(2m)^{1/3}}\left[\frac{3F(n-\frac{1}{4})\pi\hbar}{2}\right]^{2/3}}$$

For the wave fct use Eq [8.46]

$$\boxed{\begin{aligned} \psi_n(x) &= \frac{2D}{(2m)^{1/4}(E_n - Fx)^{1/4}}\sin\left[\sqrt{2m}\left(\frac{2}{3F}\right)(E_n - Fx)^{3/2}/\hbar + \frac{\pi}{4}\right] && x < E_n/F \\ &= \frac{D}{(2m)^{1/4}(Fx - E_n)^{1/4}}\exp\left\{-\sqrt{2m}\left(\frac{2}{3F}(Fx - E_n)^{3/2}\right)/\hbar\right\} && x > E_n/F \end{aligned}}$$

The wave fct at time $t$ is $|\Psi(t)\rangle = \left[e^{-iE_nt/\hbar}|\psi_n\rangle + ie^{-iE_{n+1}t/\hbar}|\psi_{n+1}\rangle\right]/\sqrt{2}$

All expectation values are the same when $(E_{n+1}-E_n)t/\hbar = 2\pi$

$$\boxed{t = \frac{2\pi\hbar}{E_{n+1}-E_n}}$$

The energies can be written as $E_n = C(n-1/4)^{2/3}$

For large $n$ $\quad E_{n+1}-E_n \cong \left.\frac{dE}{dn}\right|_{n+1/2} = \frac{2}{3}C(n+1/4)^{-1/3}$

Not asked for but interesting: This quantum period exactly equals the classical period at $E_{n+1/2}$.

Prob 7 This problem is all about identifying the states, obtaining their energies, and finding the non-zero matrix elements.

The states $\psi_{n_xn_y}(x,y) = \psi_{n_x}(x)\,\psi_{n_y}(y)$ are harmonic oscillator states with energies $E_{n_xn_y} = (n_x+n_y+1)\hbar\omega$

The energy spacings are $\boxed{\Delta E = \hbar\omega = 6.63\times10^{-19}J = 4.14\,eV}$

The only operators that can lead to a transition are $P_x$ and $yP_x$

Use Eq. [2.69] to show $P_x$ causes $\boxed{\psi_{00} \to \psi_{10}}$ and $yP_x$ causes $\boxed{\psi_{00} \to \psi_{11}}$

For the $P_x$ you need $\boxed{f_\ell = 10^{15}Hz}$ For the $yP_x$ you need $\boxed{f_\ell = 2\times10^{15}Hz}$

Calculate the matrix elements

$\langle\psi_{10}|P_x|\psi_{00}\rangle = i\sqrt{\frac{\hbar m\omega}{2}}$ $\qquad \langle\psi_{11}|yP_x|\psi_{00}\rangle = i\sqrt{\frac{\hbar m\omega}{2}}\sqrt{\frac{\hbar}{2m\omega}} = i\frac{\hbar}{2}$

For the time dependent integral $\displaystyle\int_{-\infty}^{\infty}\cos(\omega_\ell t)\,e^{i(E_b-E_a)t/\hbar}\,e^{-t^2/T^2}dt \cong \frac{\sqrt{\pi}\,T}{2}\,e^{-\frac{T^2(E_b-E_a-\hbar\omega_\ell)^2}{4\hbar^2}}$

Put together
$$\boxed{P_{10} = |A_\ell|^2\,\frac{\hbar m\omega}{2}\,\frac{\pi T^2}{4}\,e^{-T^2(\omega-\omega_\ell)^2/2}}$$
$$\boxed{P_{11} = |A_\ell|^2\,\frac{\hbar^2}{4}k^2\,\frac{\pi T^2}{4}\,e^{-T^2(2\omega-\omega_\ell)^2/2}}$$

Ratio at peaks $\boxed{P_{11}(\omega_\ell=2\omega)/P_{10}(\omega_\ell=\omega) = \frac{k^2\hbar}{2m\omega}} = \frac{4\omega^2/c^2\,\hbar}{2m\omega} = \frac{2\omega\hbar}{mc^2} = \frac{2\cdot6.626\times10^{-19}J}{8.2\times10^{-14}J} = \boxed{1.6\times10^{-5}}$