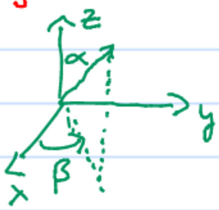
$$\langle \hat{S}_z \rangle = \chi^{+} \underline{\underline{S_z}}\, \chi = \left(\cos\left(\tfrac{\alpha}{2}\right) \;\; \sin\left(\tfrac{\alpha}{2}\right) e^{-i\beta} \right) \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} \cos\left(\frac{\alpha}{2}\right) \\ \sin\left(\frac{\alpha}{2}\right) e^{-i\beta} \end{pmatrix}$$

$$= \cos^2\left(\tfrac{\alpha}{2}\right) - \sin^2\left(\tfrac{\alpha}{2}\right) = \cos(\alpha)$$

$$\langle \hat{S}_x \rangle = \left(\cos\left(\tfrac{\alpha}{2}\right) \;\; \sin\left(\tfrac{\alpha}{2}\right) e^{-i\beta} \right) \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \cos\left(\frac{\alpha}{2}\right) \\ \sin\left(\frac{\alpha}{2}\right) e^{i\beta} \end{pmatrix}$$

$$= \cos\left(\tfrac{\alpha}{2}\right) \sin\left(\tfrac{\alpha}{2}\right) \left(e^{i\beta} + e^{-i\beta} \right) = \sin(\alpha)\cos(\beta)$$

$$\langle \hat{S}_y \rangle = \cos\left(\tfrac{\alpha}{2}\right) \sin\left(\tfrac{\alpha}{2}\right) \left(-i e^{i\beta} + i e^{-i\beta} \right) = \sin(\alpha)\sin(\beta)$$



Now add the time dependence

$$\chi(t) = e^{i\zeta} \begin{pmatrix} \cos\left(\frac{\alpha}{2}\right) e^{i\gamma B_0 t/2} \\ \sin\left(\frac{\alpha}{2}\right) e^{i\beta} e^{-i\gamma B_0 t/2} \end{pmatrix} = e^{i(\zeta + \gamma B_0 t/2)} \begin{pmatrix} \cos\left(\frac{\alpha}{2}\right) \\ \sin\left(\frac{\alpha}{2}\right) e^{i(\beta - \gamma B_0 t)} \end{pmatrix}$$

$$\langle S_z \rangle = \cos(\alpha) \qquad \langle S_x \rangle = \sin(\alpha)\cos(\beta - \gamma B_0 t) \qquad \langle S_y \rangle = \sin(\alpha)\sin(\beta - \gamma B_0 t)$$

Why doesn't $\langle S_z \rangle$ change? Why doesn't tilt change?
What does this motion look like?

Look at the book discussion of Stern Gerlach experiment.

How to write the state when there are 2 or more particles with spin?
Two particles S_1, S_2 will have joint states

$$|S_1 M_{S_1} S_2 M_{S_2}\rangle = |S_1 M_{S_1}\rangle |S_2 M_{S_2}\rangle$$

These are rarely eigenstates of $\hat{H}$

One of the more common forms of interaction is proportional to $\vec{S}_1 \cdot \vec{S}_2$

This can be written as $\vec{S}_1 \cdot \vec{S}_2 = \frac{1}{2}\left[\left(\vec{S}_1 + \vec{S}_2 \right)^2 - \vec{S}_1^{\,2} - \vec{S}_2^{\,2} \right]$

Instead of using eigenstates of $\vec{S}_1^2, S_{1z}, \vec{S}_2^2, S_{2z}$ use the eigenstates of $\vec{S}_1^2, \vec{S}_2^2, \vec{S}^2, S_z$

$\vec{S} \equiv \vec{S}_1 + \vec{S}_2$ $\qquad$ $S_z = S_{1z} + S_{2z}$

In classical mechanics, the $\vec{S}_1$ and $\vec{S}_2$ can be in any direction. The $\max|\vec{S}_1+\vec{S}_2| = |\vec{S}_1|+|\vec{S}_2|$ and the $\min|\vec{S}_1+\vec{S}_2| = \big||\vec{S}_1|-|\vec{S}_2|\big|$

In Q.M. the allowed eigenvalues of $\vec{S}^2$ are $\hbar^2 s(s+1)$ with $s_1+s_2,\ s_1+s_2-1, \ldots, |s_1-s_2|$

Example: Spin $\frac{5}{2}$ and spin $\frac{3}{2}$ Count states

Total number of $\frac{5}{2}$ states $2\,\frac{5}{2}+1 = 6$

" " " $\frac{3}{2}$ " $2\,\frac{3}{2}+1 = 4$

" " " states $= 6\cdot 4 = 24$

Total number of $\frac{5}{2}+\frac{3}{2}=4$ states $2\cdot 4+1 = 9$

" " " $\frac{5}{2}+\frac{3}{2}-1=3$ " $2\cdot 3+1=7$

" " " $\frac{5}{2}+\frac{3}{2}-2=2$ " $2\cdot 2+1=5$

" " " $\frac{5}{2}-\frac{3}{2}=1$ " $2\cdot 1+1=3$

sum = 24 states

Strategy for finding all of the eigenstates of $\vec{S}^2$ and S_z

① The state with $m_{s_1}=s_1$ and $m_{s_2}=s_2$ must be $s=s_1+s_2$, $m_s=s$

② Act on that state with $\hat{S}_-/(\hbar A_{s m_s})$. This must be $s=s_1+s_2$, $m_s=s-1$

③ Do $\hat{S}_-/(\hbar A_{s m_s-1})$ This must be $s=s_1+s_2$, $m_s=s-2$

etc until you've got $s=s_1+s_2$, $m_s=-s$

(?) The $s=s_1+s_2-1$, $m_s=s_1+s_2-1$ state must be the state orthogonal to the $s=s_1+s_2$, $m_s=s_1+s_2-1$ state

(?+1) Apply the $\hat{S}_-/(\hbar A_{s m_s})$ to this state to get $s=s_1+s_2-1$, $m_s=s_1+s_2-2$

etc

(??) The $s=s_1+s_2-2$, $m_s=s_1+s_2-2$ must be orthogonal to the states $s=s_1+s_2$, $m_s=s_1+s_2-2$ and $s=s_1+s_2-1$, $m_s=s_1+s_2-2$ states.

etc

Example: Spin 1 and spin $\frac{1}{2}$

$$|\tfrac{3}{2},\tfrac{3}{2}\rangle = |1,1\rangle|\tfrac{1}{2},\tfrac{1}{2}\rangle$$

$$|\tfrac{3}{2},\tfrac{1}{2}\rangle = \tfrac{1}{\hbar}(S_{1-} + S_{2-})|1,1\rangle|\tfrac{1}{2},\tfrac{1}{2}\rangle \Big/ \sqrt{\left(\tfrac{15}{4} - \tfrac{3}{4}\right)} = \sqrt{\tfrac{2}{3}}|1,0\rangle|\tfrac{1}{2},\tfrac{1}{2}\rangle + \sqrt{\tfrac{1}{3}}|1,1\rangle|\tfrac{1}{2},-\tfrac{1}{2}\rangle$$

$\vdots$

$$|\tfrac{1}{2},\tfrac{1}{2}\rangle = -\sqrt{\tfrac{1}{3}}|1,0\rangle|\tfrac{1}{2},\tfrac{1}{2}\rangle + \sqrt{\tfrac{2}{3}}|1,1\rangle|\tfrac{1}{2},-\tfrac{1}{2}\rangle$$

You can look up these coefficients

$$|S,M_S\rangle = \sum_{M_{1S}M_{2S}} |S_1 M_{1S}\rangle|S_2 M_{2S}\rangle \langle S_1 M_{1S} S_2 M_{2S}|S M_S\rangle \qquad M_{1S} + M_{2S} = M_S$$

The $\langle j_1 m_1 j_2 m_2 | j m\rangle$ are Clebsch Gordon coefficients

Book notation $C^{s_1 s_2 s}_{m_1 m_2 m} = \langle s_1 m_1 s_2 m_2 | s m\rangle$

Show how to read the C.G. table

Example: $S_1 = \frac{3}{2}$ $S_2 = 1$

$$|\tfrac{5}{2},\tfrac{5}{2}\rangle = |\tfrac{3}{2},\tfrac{3}{2}\rangle|1,1\rangle$$

$$|\tfrac{5}{2},\tfrac{3}{2}\rangle = \sqrt{\tfrac{2}{5}}|\tfrac{3}{2},\tfrac{3}{2}\rangle|1,0\rangle + \sqrt{\tfrac{3}{5}}|\tfrac{3}{2},\tfrac{1}{2}\rangle|1,1\rangle$$

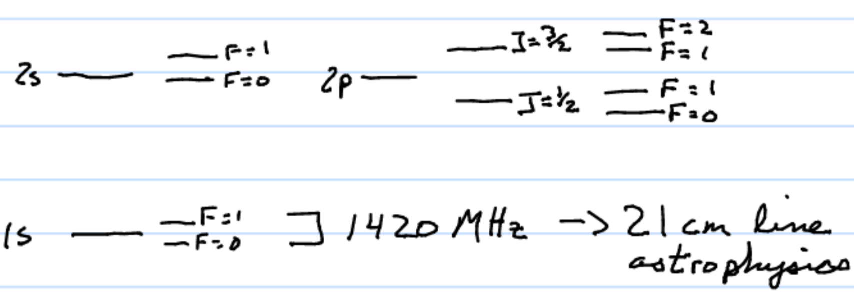
$$|\tfrac{3}{2},\tfrac{3}{2}\rangle = \sqrt{\tfrac{3}{5}}|\tfrac{3}{2},\tfrac{3}{2}\rangle|1,0\rangle - \sqrt{\tfrac{2}{5}}|\tfrac{3}{2},\tfrac{1}{2}\rangle|1,1\rangle$$

$$|\tfrac{5}{2},\tfrac{1}{2}\rangle = \sqrt{\tfrac{1}{10}}|\tfrac{3}{2},\tfrac{3}{2}\rangle|1,-1\rangle + \sqrt{\tfrac{6}{10}}|\tfrac{3}{2},\tfrac{1}{2}\rangle|1,0\rangle + \sqrt{\tfrac{3}{10}}|\tfrac{3}{2},-\tfrac{1}{2}\rangle|1,1\rangle$$

$$|\tfrac{3}{2},\tfrac{1}{2}\rangle = \sqrt{\tfrac{6}{15}}|\tfrac{3}{2},\tfrac{3}{2}\rangle|1,-1\rangle + \sqrt{\tfrac{1}{15}}|\tfrac{3}{2},\tfrac{1}{2}\rangle|1,0\rangle - \sqrt{\tfrac{8}{15}}|\tfrac{3}{2},-\tfrac{1}{2}\rangle|1,1\rangle$$

$$|\tfrac{1}{2},\tfrac{1}{2}\rangle = \sqrt{\tfrac{3}{6}}|\tfrac{3}{2},\tfrac{3}{2}\rangle|1,-1\rangle - \sqrt{\tfrac{2}{6}}|\tfrac{3}{2},\tfrac{1}{2}\rangle|1,0\rangle + \sqrt{\tfrac{1}{6}}|\tfrac{3}{2},-\tfrac{1}{2}\rangle|1,1\rangle$$

Schematic of fine structure and hyperfine structure in H



$\vec{J} = \vec{L} + \vec{S}_e$ Fine

$\vec{F} = \vec{J} + \vec{S}_p$ Hyperfine

Example: Energy splittings when $\hat{H} = \varepsilon_0 \vec{S}_1 \cdot \vec{S}_2 / \hbar^2$

$$\vec{S}_1 \cdot \vec{S}_2 = \frac{1}{2}\left(\vec{S}^2 - \vec{S}_1^2 - \vec{S}_2^2 \right)$$

$$E_s = \frac{\varepsilon_0}{2}\left[s(s+1) - s_1(s_1+1) - s_2(s_2+1) \right]$$

$s = |s_1 - s_2|, |s_1 - s_2| + 1, \ldots$
$\ldots, s_1 + s_2 - 1, s_1 + s_2$

For H ground state $s = 1$ and 0

$$E_1 - E_0 = (\varepsilon_0/2)\left[1\cdot 2 - \tfrac{3}{4} - \tfrac{3}{4}\right] - (\varepsilon_0/2)\left[0\cdot 1 - \tfrac{3}{4} - \tfrac{3}{4}\right] = \varepsilon_0 = h \cdot 1420\,\mathrm{MHz}$$

34. CLEBSCH-GORDAN COEFFICIENTS, SPHERICAL HARMONICS, AND d FUNCTIONS

Note: A square-root sign is to be understood over *every* coefficient, *e.g.*, for −8/15 read $-\sqrt{8/15}$.

Notation:

	J	J	...
	M	M	...
m_1 m_2	Coefficients		
m_1 m_2			
. .			
. .			

$$Y_1^0 = \sqrt{\frac{3}{4\pi}}\cos\theta$$

$$Y_1^1 = -\sqrt{\frac{3}{8\pi}}\sin\theta\, e^{i\phi}$$

$$Y_2^0 = \sqrt{\frac{5}{4\pi}}\Big(\frac{3}{2}\cos^2\theta - \frac{1}{2}\Big)$$

$$Y_2^1 = -\sqrt{\frac{15}{8\pi}}\sin\theta\cos\theta\, e^{i\phi}$$

$$Y_2^2 = \frac{1}{4}\sqrt{\frac{15}{2\pi}}\sin^2\theta\, e^{2i\phi}$$

$$Y_\ell^{-m} = (-1)^m Y_\ell^{m*}$$

$$d^{\,\ell}_{m,0} = \sqrt{\frac{4\pi}{2\ell+1}}\, Y_\ell^m\, e^{-im\phi}$$

$$\langle j_1 j_2 m_1 m_2 | j_1 j_2 J M\rangle = (-1)^{J-j_1-j_2}\langle j_2 j_1 m_2 m_1 | j_2 j_1 J M\rangle$$

1/2×1/2

	1
	+1
+1/2 +1/2	1

	1	0
	0	0
+1/2 −1/2	1/2	1/2
−1/2 +1/2	1/2	−1/2

	1
	−1
−1/2 −1/2	1

1×1/2

	3/2
	+3/2
+1 +1/2	1

	3/2	1/2
	+1/2	+1/2
+1 −1/2	1/3	2/3
0 +1/2	2/3	−1/3

	3/2	1/2
	−1/2	−1/2
0 −1/2	2/3	1/3
−1 +1/2	1/3	−2/3

	3/2
	−3/2
−1 −1/2	1

2×1/2

	5/2
	+5/2
+2 +1/2	1

	5/2	3/2
	+3/2	+3/2
+2 −1/2	1/5	4/5
+1 +1/2	4/5	−1/5

	5/2	3/2
	+1/2	+1/2
+1 −1/2	2/5	3/5
0 +1/2	3/5	−2/5

	5/2	3/2
	−1/2	−1/2
0 −1/2	3/5	2/5
−1 +1/2	2/5	−3/5

	5/2	3/2
	−3/2	−3/2
−1 −1/2	4/5	1/5
−2 +1/2	1/5	−4/5

	5/2
	−5/2
−2 −1/2	1

3/2×1/2

	2
	+2
+3/2 +1/2	1

	2	1
	+1	+1
+3/2 −1/2	1/4	3/4
+1/2 +1/2	3/4	−1/4

	2	1
	0	0
+1/2 −1/2	1/2	1/2
−1/2 +1/2	1/2	−1/2

	2	1
	−1	−1
−1/2 −1/2	3/4	1/4
−3/2 +1/2	1/4	−3/4

	2
	−2
−3/2 −1/2	1

2×1

	3
	+3
+2 +1	1

	3	2
	+2	+2
+2 0	1/3	2/3
+1 +1	2/3	−1/3

	3	2	1
	+1	+1	+1
+2 −1	1/15	1/3	3/5
+1 0	8/15	1/6	−3/10
0 +1	2/5	−1/2	1/10

	3	2	1
	0	0	0
+1 −1	1/5	1/2	3/10
0 0	3/5	0	−2/5
−1 +1	1/5	−1/2	3/10

	3	2	1
	−1	−1	−1
0 −1	2/5	1/2	1/10
−1 0	8/15	−1/6	−3/10
−2 +1	1/15	−1/3	3/5

	3	2
	−2	−2
−1 −1	2/3	1/3
−2 0	1/3	−2/3

	3
	−3
−2 −1	1

1×1

	2
	+2
+1 +1	1

	2	1
	+1	+1
+1 0	1/2	1/2
0 +1	1/2	−1/2

	2	1	0
	0	0	0
+1 −1	1/6	1/2	1/3
0 0	2/3	0	−1/3
−1 +1	1/6	−1/2	1/3

	2	1
	−1	−1
0 −1	1/2	1/2
−1 0	1/2	−1/2

	2
	−2
−1 −1	1

3/2×1

	5/2
	+5/2
+3/2 +1	1

	5/2	3/2
	+3/2	+3/2
+3/2 0	2/5	3/5
+1/2 +1	3/5	−2/5

	5/2	3/2	1/2
	+1/2	+1/2	+1/2
+3/2 −1	1/10	2/5	1/2
+1/2 0	3/5	1/15	−1/3
−1/2 +1	3/10	−8/15	1/6

	5/2	3/2	1/2
	−1/2	−1/2	−1/2
+1/2 −1	3/10	8/15	1/6
−1/2 0	3/5	−1/15	−1/3
−3/2 +1	1/10	−2/5	1/2

	5/2	3/2
	−3/2	−3/2
−1/2 −1	3/5	2/5
−3/2 0	2/5	−3/5

	5/2
	−5/2
−3/2 −1	1

$$d^{\,j}_{m',m} = (-1)^{m-m'} d^{\,j}_{m,m'} = d^{\,j}_{-m,-m'}$$

3/2×3/2

	3
	+3
+3/2 +3/2	1

	3	2
	+2	+2
+3/2 +1/2	1/2	1/2
+1/2 +3/2	1/2	−1/2

	3	2	1
	+1	+1	+1
+3/2 −1/2	1/5	1/2	3/10
+1/2 +1/2	3/5	0	−2/5
−1/2 +3/2	1/5	−1/2	3/10

	3	2	1	0
	0	0	0	0
+3/2 −3/2	1/20	1/4	9/20	1/4
+1/2 −1/2	9/20	1/4	−1/20	−1/4
−1/2 +1/2	9/20	−1/4	−1/20	1/4
−3/2 +3/2	1/20	−1/4	9/20	−1/4

	3	2	1
	−1	−1	−1
+1/2 −3/2	1/5	1/2	3/10
−1/2 −1/2	3/5	0	−2/5
−3/2 +1/2	1/5	−1/2	3/10

	3	2
	−2	−2
−1/2 −3/2	1/2	1/2
−3/2 −1/2	1/2	−1/2

	3
	−3
−3/2 −3/2	1

$$d^{\,1}_{0,0} = \cos\theta \qquad d^{\,1/2}_{1/2,1/2} = \cos\frac{\theta}{2} \qquad d^{\,1}_{1,1} = \frac{1+\cos\theta}{2}$$

$$d^{\,1/2}_{1/2,-1/2} = -\sin\frac{\theta}{2} \qquad d^{\,1}_{1,0} = -\frac{\sin\theta}{\sqrt{2}}$$

$$d^{\,1}_{1,-1} = \frac{1-\cos\theta}{2}$$

2×3/2

	7/2
	+7/2
+2 +3/2	1

	7/2	5/2
	+5/2	+5/2
+2 +1/2	3/7	4/7
+1 +3/2	4/7	−3/7

	7/2	5/2	3/2
	+3/2	+3/2	+3/2
+2 −1/2	1/7	16/35	2/5
+1 +1/2	4/7	1/35	−2/5
0 +3/2	2/7	−18/35	1/5

	7/2	5/2	3/2	1/2
	+1/2	+1/2	+1/2	+1/2
+2 −3/2	1/35	6/35	2/5	2/5
+1 −1/2	12/35	5/14	0	−3/10
0 +1/2	18/35	−3/35	−1/5	1/5
−1 +3/2	4/35	−27/70	2/5	−1/10

	7/2	5/2	3/2	1/2
	−1/2	−1/2	−1/2	−1/2
+1 −3/2	4/35	27/70	2/5	1/10
0 −1/2	18/35	3/35	−1/5	−1/5
−1 +1/2	12/35	−5/14	0	3/10
−2 +3/2	1/35	−6/35	2/5	−2/5

	7/2	5/2	3/2
	−3/2	−3/2	−3/2
0 −3/2	2/7	18/35	1/5
−1 −1/2	4/7	−1/35	−2/5
−2 +1/2	1/7	−16/35	2/5

	7/2	5/2
	−5/2	−5/2
−1 −3/2	4/7	3/7
−2 −1/2	3/7	−4/7

	7/2
	−7/2
−2 −3/2	1

2×2

	4
	+4
+2 +2	1

	4	3
	+3	+3
+2 +1	1/2	1/2
+1 +2	1/2	−1/2

	4	3	2
	+2	+2	+2
+2 0	3/14	1/2	2/7
+1 +1	4/7	0	−3/7
0 +2	3/14	−1/2	2/7

	4	3	2	1
	+1	+1	+1	+1
+2 −1	1/14	3/10	3/7	1/5
+1 0	3/7	1/5	−1/14	−3/10
0 +1	3/7	−1/5	−1/14	3/10
−1 +2	1/14	−3/10	3/7	−1/5

	4	3	2	1	0
	0	0	0	0	0
+2 −2	1/70	1/10	2/7	2/5	1/5
+1 −1	8/35	2/5	1/14	−1/10	−1/5
0 0	18/35	0	−2/7	0	1/5
−1 +1	8/35	−2/5	1/14	1/10	−1/5
−2 +2	1/70	−1/10	2/7	−2/5	1/5

	4	3	2	1
	−1	−1	−1	−1
+1 −2	1/14	3/10	3/7	1/5
0 −1	3/7	1/5	−1/14	−3/10
−1 0	3/7	−1/5	−1/14	3/10
−2 +1	1/14	−3/10	3/7	−1/5

	4	3	2
	−2	−2	−2
0 −2	3/14	1/2	2/7
−1 −1	4/7	0	−3/7
−2 0	3/14	−1/2	2/7

	4	3
	−3	−3
−1 −2	1/2	1/2
−2 −1	1/2	−1/2

	4
	−4
−2 −2	1

$$d^{\,3/2}_{3/2,3/2} = \frac{1+\cos\theta}{2}\cos\frac{\theta}{2}$$

$$d^{\,3/2}_{3/2,1/2} = -\sqrt{3}\frac{1+\cos\theta}{2}\sin\frac{\theta}{2} \qquad d^{\,2}_{2,2} = \Big(\frac{1+\cos\theta}{2}\Big)^2$$

$$d^{\,3/2}_{3/2,-1/2} = \sqrt{3}\frac{1-\cos\theta}{2}\cos\frac{\theta}{2} \qquad d^{\,2}_{2,1} = -\frac{1+\cos\theta}{2}\sin\theta$$

$$d^{\,3/2}_{3/2,-3/2} = -\frac{1-\cos\theta}{2}\sin\frac{\theta}{2} \qquad d^{\,2}_{2,0} = \frac{\sqrt{6}}{4}\sin^2\theta \qquad d^{\,2}_{1,1} = \frac{1+\cos\theta}{2}(2\cos\theta-1)$$

$$d^{\,3/2}_{1/2,1/2} = \frac{3\cos\theta-1}{2}\cos\frac{\theta}{2} \qquad d^{\,2}_{2,-1} = -\frac{1-\cos\theta}{2}\sin\theta \qquad d^{\,2}_{1,0} = -\sqrt{\frac{3}{2}}\sin\theta\cos\theta$$

$$d^{\,3/2}_{1/2,-1/2} = -\frac{3\cos\theta+1}{2}\sin\frac{\theta}{2} \qquad d^{\,2}_{2,-2} = \Big(\frac{1-\cos\theta}{2}\Big)^2 \qquad d^{\,2}_{1,-1} = \frac{1-\cos\theta}{2}(2\cos\theta+1) \qquad d^{\,2}_{0,0} = \Big(\frac{3}{2}\cos^2\theta - \frac{1}{2}\Big)$$

Figure 34.1: The sign convention is that of Wigner (*Group Theory*, Academic Press, New York, 1959), also used by Condon and Shortley (*The Theory of Atomic Spectra*, Cambridge Univ. Press, New York, 1953), Rose (*Elementary Theory of Angular Momentum*, Wiley, New York, 1957), and Cohen (*Tables of the Clebsch-Gordan Coefficients*, North American Rockwell Science Center, Thousand Oaks, Calif., 1974). The coefficients here have been calculated using computer programs written independently by Cohen and at LBNL.